Department of Physics and Astronomy, University of New Mexico

E&M Preliminary Examination

Spring 2012

Instructions:

- The exam consists of 10 problems (10 pts each).
- Partial credit will be given if merited.
- Personal notes on two sides of an 8.5"x 11" sheet are allowed.
- Total time: 3 hours.

Possibly Useful Formulas

- Azimuthally symmetric (ϕ-independent) solution of the Laplace equation in spherical polar coordinates:

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta),$$

  where the first few Legendre polynomials are defined as

$$P_0(\mu) = 1; \quad P_1(\mu) = \mu; \quad P_2(\mu) = \frac{1}{2}(3\mu^2 - 1); \quad P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu); \quad \text{etc.}$$

- The electric potential at position $\vec{r}$ due to a point electric dipole of moment $\vec{p}$ located at position $\vec{r}'$:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}.$$

- Magnetic field at position $\vec{r}$ due to a point magnetic dipole of moment $\vec{m}$ located at the origin:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \left[\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3} \right].$$

- Biot-Savart Law for the magnetic field at position $\vec{r}$ due to a steady current element $Id\vec{\ell}'$ located at position $\vec{r}'$:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{Id\vec{\ell}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}.$$

- Time-averaged power radiated by an oscillating electric dipole:

$$P = \frac{\mu_0 |p|^2 \omega^4}{12\pi c}.$$

- Time-averaged power radiated by an oscillating magnetic dipole:

$$P = \frac{\mu_0 |m|^2 \omega^4}{12\pi c^3}.$$

- Instantaneous power radiated by a non-relativistically moving charge with acceleration a (Larmor formula):

$$P = \frac{q^2 a^2}{6\pi \epsilon_0 c^3}.$$

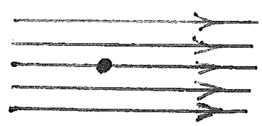
- The electromagnetic field transforms as follows under Lorentz transformation:

$$E'_{\parallel} = E_{\parallel}, \quad B'_{\parallel} = B_{\parallel},$$

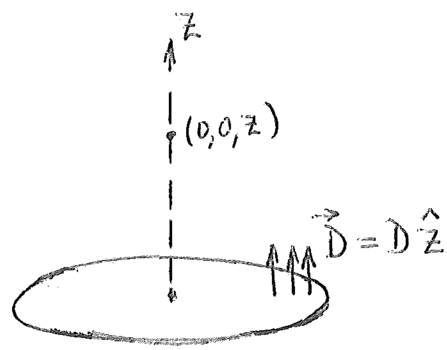
$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}), \quad \vec{B}'_{\perp} = \gamma \left(\vec{B}_{\perp} - \frac{\vec{v}}{c^2} \times \vec{E} \right),$$

  where the primed frame is moving at velocity $\vec{v}$ relative to the unprimed frame, and the subscripts $\parallel$ and $\perp$ indicate components parallel and perpendicular to the direction of relative velocity $\vec{v}$, respectively. The symbol γ denotes the usual Lorentz contraction factor, $\gamma = 1/\sqrt{1 - v^2/c^2}$.

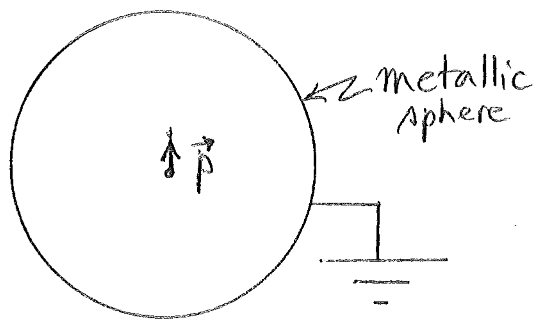
1. A point charge q of mass m is accelerated in a uniform electric field along the x axis, $\vec{E} = E\hat{x}$. Before the field is applied, the charge is at rest at the origin. Under fully relativistic motion, show that the acceleration at any time during the motion is equal to its initial value divided by γ^3, where γ denotes the usual Lorentz factor. How fast is the charge moving at the time its acceleration is 1/8th of its initial value? (*Hint:* Use the work-energy theorem that equates the rate of change of kinetic energy to the rate of work done.)



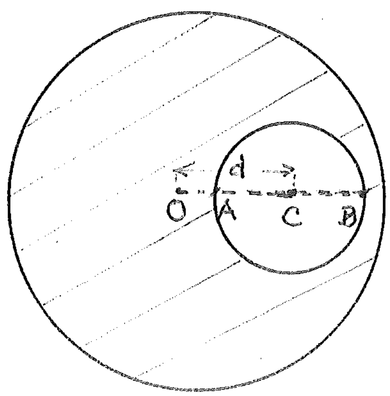
2. A thin circular dielectric disk of radius R is uniformly polarized along a direction normal to its plane, so the electric dipole moment per unit area is $D\hat{z}$. Determine the potential everywhere on the axis of the disk as a function of the distance from the disk on both sides. Show that this potential jumps discontinuously by an amount D/ϵ_0 from one side of the disk to the other.



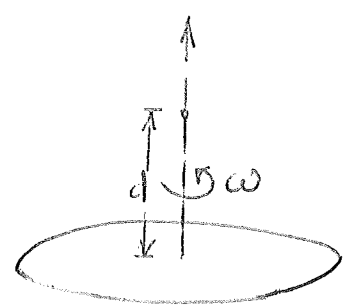
3. A point electric dipole of moment $p\hat{z}$ is located at the center of an electricaly grounded, hollow metallic sphere of radius R. Express the electric potential due to the dipole alone at any point $\vec{r}$ in the space enclosed by the sphere, using spherical coordinates r and θ. The dipole field induces charges on the grounded metallic sphere, which contribute the rest of the potential at any point. Argue why this contribution to the potential must have the form $Ar\cos\theta$, where A is a constant. By requiring that the total potential of the surface of the sphere ~~is~~ be zero, calculate the constant A in terms of p, R, and ϵ_0.



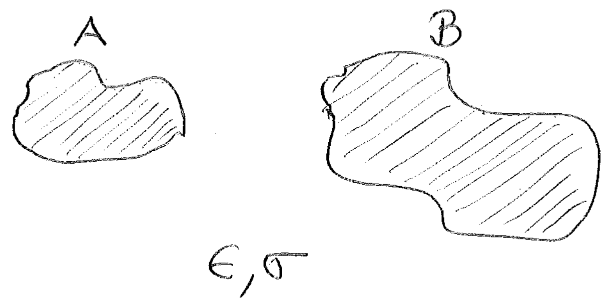
4. A steady current I flows inside a long cylindrical solid wire of radius a. The current is uniformly distributed in the cross-section of the wire. What are the magnitude and direction of the magnetic field everywhere inside the wire? If a cylindrical hole of radius b ($b < a/2$) is drilled in the original wire parallel to its axis at a center-to-center distance of d, with $d < (a - b)$, without disturbing the current flowing in the rest of the wire, then what are the magnitude and direction of the magnetic field at the center (C) of the hole? At the innermost and outermost points (A and B) of the hole?



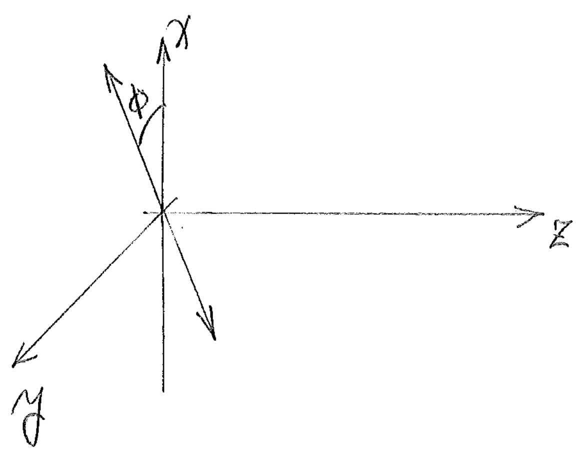
5. A thin, uniformly charged disk of radius R and charge Q is rotated at a uniform angular velocity ω about its axis. Calculate the magnitude and direction of the magnetic field due to the rotating disk at a distance d from it along its axis.



6. Two metal objects are embedded in a weakly conducting, linear dielectric medium of permittivity ϵ and conductivity σ. If the capacitance of the system is C, then show that the resistance of the system is given by $R = \epsilon/(\sigma C)$. (*Hint:* Put charges $\pm Q$ on the two objects and then consider the flux of the electric field through a Gaussian surface surrounding one of them. Can you then relate the electric flux to the total current passing between the objects?)



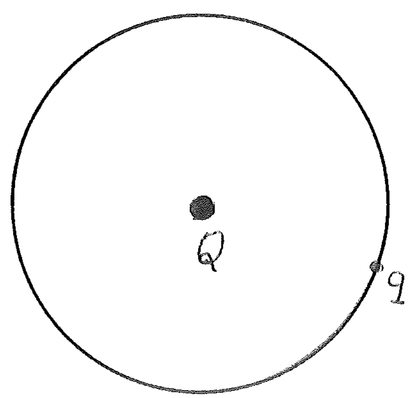
7. A monochromatic plane wave of light propagating along $\hat{z}$ and of angular frequency ω is linearly polarized in a plane oriented at angle ϕ to the xz plane. Express the electric field of the wave as a superposition of two circularly polarized waves. What is the phase difference between the two circularly polarized waves? If the plane wave is now allowed to pass inside a highly transparent but optically active medium of length L with two different indices $n_{\pm}$ for the two opposite-helicity circular polarizations, then what would be the nature of polarization of the wave upon emerging from the medium? Be quantitatively explicit in your answer.



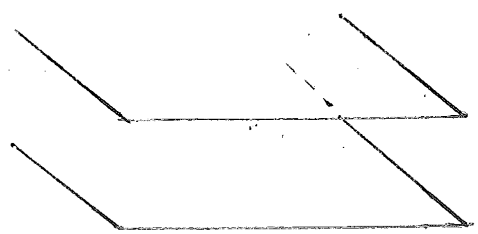
8. A small charged body of charge q and mass m drifting uniformly through space finds itself trapped at time $t = 0$ in a circular orbit around a uniformly charged, highly massive spherical body of total charge Q (of opposite sign relative to q). What are the kinetic energy, potential energy, and total mechanical energy of the orbiting body when its orbital radius is r? Because of its acceleration, the orbiting charged body radiates energy. Assuming its motion to be non-relativistic, calculate using Larmor's formula the rate at which it radiates energy. Using energy conservation and assuming that the fractional loss of mechanical energy per orbit is very small compared to 1, show that the body will lose its orbital radius as a function of time via the relation

$$r^3(t) = r^3(0) - \alpha t,$$

where α is a positive constant. Express α in terms of q, Q, m, c, and ϵ_0.



9. A monochromatic electromagnetic (EM) wave of angular frequency ω propagates in the free space between two infinitely extended, plane metal plates. Show that this parallel-plate waveguide supports a TEM mode. By using the EM boundary conditions, determine the directions of the electric and magnetic fields in this mode. By means of the Maxwell equations, show that the ratio of the magnitudes of the electric and magnetic field is equal to c for this mode. What other types of guided monochromatic modes are possible in this waveguide? Sketch the EM field configuration in one such mode.



10. A parallel plate capacitor, at rest in frame S' and tilted at 45^0 relative to the x-axis in that frame, carries surface charge densities $\pm\sigma'$on its plates. The frame S' is moving at speed v along the x axis relative to the laboratory frame S. What are the electric and magnetic fields in frame S'? In frame S? At what angle are the plates tilted in frame S? Is the electric field perpendicular to the plates in frame S?