Department of Physics and Astronomy, University of New Mexico

E&M Preliminary Examination

Spring 2011

Instructions:

- The exam consists of 10 problems (10 pts each).
- Partial credit will be given if merited.
- Personal notes on two sides of an 8.5"x 11" sheet are allowed.
- Total time: 3 hours.

Possibly Useful Formulas

- Relation of spherical polar coordinates, (r, θ, ϕ), to Cartesian coordinates:

$$x = r\,\sin\theta\,\cos\phi, \quad y = r\,\sin\theta\,\sin\phi, \quad z = r\,\cos\theta.$$

Unit vectors:

$$\hat{r} = \sin\theta\,\cos\phi\,\hat{x} + \sin\theta\,\sin\phi\,\hat{y} + \cos\theta\,\hat{z};$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}; \quad \hat{\theta} = \hat{\phi}\times\hat{r}.$$

- Azimuthally symmetric (ϕ-independent) solution of the Laplace equation in spherical polar coordinates:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell\, r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta),$$

where the first few Legendre polynomials are defined as

$$P_0(\cos\theta) = 1; \quad P_1(\cos\theta) = \cos\theta; \quad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1); \quad \text{etc.}$$

- Electric field at position $\vec{r}$ due to a point electric dipole of moment $\vec{p}$ located at the origin:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3}\right].$$

- Biot-Savart Law for the magnetic field at position $\vec{r}$ due to a steady current element $I\vec{d\ell}'$ located at position $\vec{r}'$:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I\vec{d\ell}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}.$$

- Force on a magnetic dipole $\vec{m}$ in a nonuniform external magnetic field $\vec{B}$:

$$\vec{F} = (\vec{m} \cdot \vec{\nabla})\vec{B}.$$

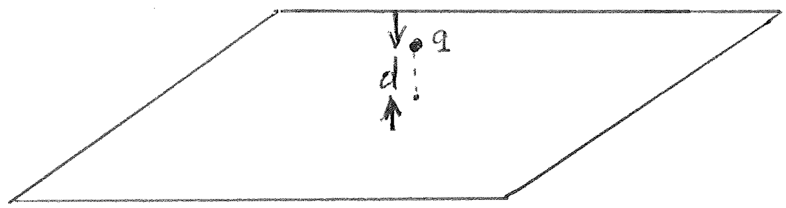
- Power radiated by an oscillating electric dipole:

$$P = \frac{\mu_0 |p|^2 \omega^4}{12\pi c}.$$

- Power radiated by an oscillating magnetic dipole:

$$P = \frac{\mu_0 |m|^2 \omega^4}{12\pi c^3}.$$

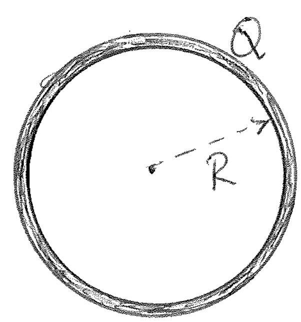
1. A large amount of charge, say Q, is placed on a large square plane sheet of area A. Take the limits, $Q \to \infty$, $A \to \infty$, such that $\sigma = Q/A$ is finite.



   (a) What are the magntiude and direction of the electric field due to the charged plane on either side of it a distance d away?

   (b) A small point charge q of the same sign as Q and mass m is released from rest at a distance d from the charged plane. Describe its position and velocity as a function of time t for sufficiently small t. How large can t be for your simple answer to be approximately correct non-relativistically?

2. A thin spherical shell of radius R, made of an insulating material, is charged uniformly so the total charge on it is Q.



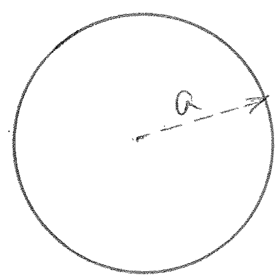
   (a) What are the magnitude and direction of the electric field at the center of the space enclosed by the shell? At a distance $d < R$ from the center?

(b) What are the magnitude and direction of the electric field outside the sphere a distance $d > R$ from its center?

(c) If a point electric dipole of moment $\vec{p} = p\hat{z}$ is placed at the center of the sphere, what is the electric field outside the sphere at $\vec{r} = d\hat{z}$ $(d > R)$?

(d) If the shell were perfectly conducting, would the answer to your previous part change? How and why?

3. A surface charge distribution situated on a hollow spherical shell of radius a produces a spatially non-uniform electric field of form, $\vec{E} = (E_0/a)(x, y, -2z)$, in the volume enclosed by the shell, *i.e.*, for $r < a$.

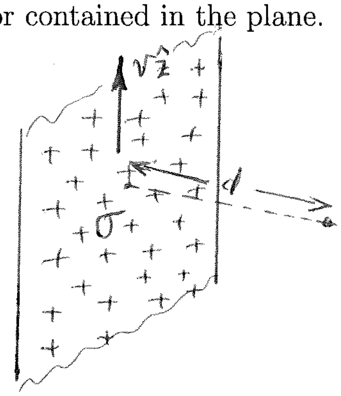


(a) What is the electric potential, $\Phi(x, y, z)$, corresponding to this electric field?

(b) Express $\Phi(x, y, z)$ in spherical polar coordinates, identifying the angular dependence as a low-order Legendre polynomial. Ignore any overall additive constant in Φ.

(c) What angular form of the surface charge distribution on the shell could produce such a potential?

4. An infinitely extended plane possessing a uniform surface charge density σ is moving at a uniform velocity $\vec{v} = v\hat{z}$, where $\hat{z}$ is a constant unit vector contained in the plane.

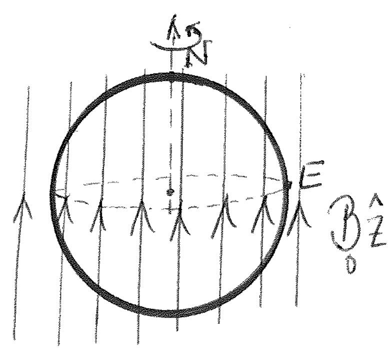


(a) What are the magnitude and direction of the surface current density, $\vec{K}$, in the plane?

(b) What are the magnitude and direction of the magnetic and electric fields a distance d on either side of the plane?

(c) Could a test charge move in the empty space outside the plane without any net electromagnetic acceleration? If so, what are the magnitude and direction of the charge velocity for which this could happen. If not, explain why not.

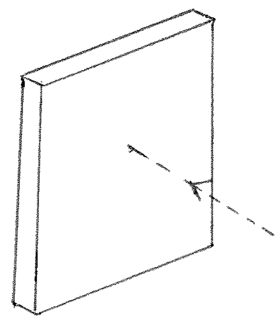
5. A perfectly conducting shell of radius a rotates about the z axis with angular velocity ω. There exists a uniform magnetic field $B_0\hat{z}$ everywhere.



(a) In steady state, there must be an electric field in the material of the conducting shell. Why?

(b) Calculate the motional EMF induced between the north pole (N) and a point E on the equator.

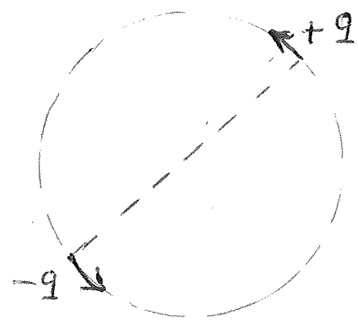
6. A thin transparent glass slab is oriented normally to an incident electromagnetic plane wave of time averaged intensity I. Let its back surface be fully silvered so it completely reflects all the incident radiation.



(a) What is time-averaged force of radiation if the exposed front surface of the slab has area A?

(b) If the silvering were to be removed so only a fraction R ($R < 1$) of the incident power is reflected and the rest transmitted, then what is the new radiation force on the slab?

7. Two point masses of equal mass m and equal but oppositely signed charges $\pm q$ are bound in uniform circular motion of radius R around a common center due to their Coulomb attraction. Express the speed and angular frequency of circulation of each charge about the center in terms of q, m, R, and fundamental constants. Under what condition involving these quantities will the charges radiate predominantly electric-dipole (ED) radiation? Show that the power radiated by the charges in the ED limit is inversely proportional to R^4. Why does the charge pair not emit any magnetic dipole radiation at all? (No detailed expressions for the radiated fields are needed here.)



8. A straight wire of conductivity σ has a circular cross-section of radius a. It carries a steady current I. Calculate the magnitude and direction of the Poynting vector at the surface of the wire. What is the rate at which Joule heat is produced in a segment of length L of the wire? Show that these two results are consistent with the conservation of energy.

9. An incident EM pulse of center frequency ω propagates inside a plasma. The dielectric constant of the plasma at frequency ω is $1-\omega_p^2/\omega^2$, where ω_p is its plasma frequency. Take $\omega > \omega_p$. What is the phase velocity of the EM pulse inside the plasma? What is its group velocity? Show that the group velocity is always smaller than c while the phase velocity is always greater than c, but their product is always c^2. Will the EM pulse maintain its length as it propagates inside the plasma? Why or why not?

10. Consider a thin conducting shell of radius R carrying a surface current density $\vec{K}$ that is directed azimuthally about a diameter and has a fixed magnitude K everywhere. Calculate the total magnetic dipole moment of the current shell. If it is now rotated rigidly so that diameter rotates about the center of the shell in a plane at angular frequency ω, then show that for sufficiently small radius, R, the current shell will radiate magnetic dipole radiation. Argue why the radiated power must scale with radius as R^6. (No detailed expressions for the radiated fields are needed here.)