Preliminary Examination: Electricity and Magnetism

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Spring 2005

Instructions:

- The exam consists of two parts: 5 short answers (6 points each) and your choice of 2 out of 3 long answer problems (35 points each).
- Where possible show ALL work and partial credit will be given.
- Personal notes on 2 sides of one 8" × 11" page are allowed.
- Total time: 3 hours

Best wishes!!

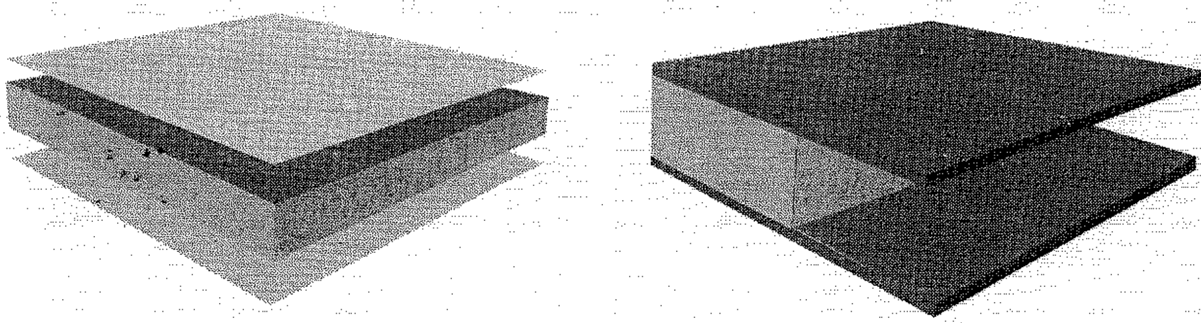
Short problems:

S1: Suppose that you have enough linear dielectric material, of dielectric constant ϵ_r, to *half*-fill a parallel-plate capacitor. Your goal is to obtain the maximum capacitance. For purpose of the calculation, A is the area of the capacitor plates, d is the distance between the plates and $d <<< A^{0.5}$. Thus as usual ignore fringe field effects.

(a) What is the capacitance for configuration (a) [see figure below]?

(b) What is the capacitance for configuration (b)?

(c) So which design is best?

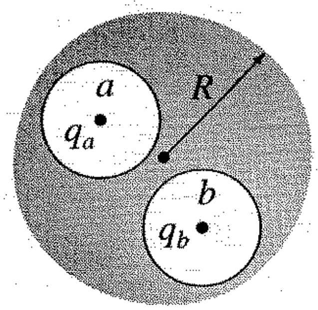


(a) (b)

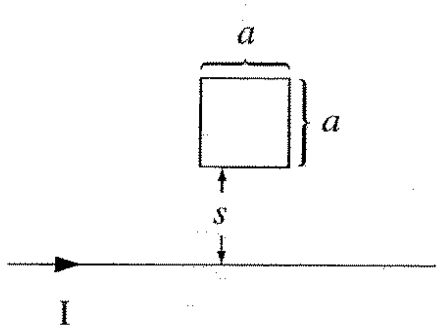
S2: Two spherical cavities, of radii "a" and "b", are hollowed out from the interior of a (neutral) conducting sphere of radius "R" [see figure]. At the center of each cavity a point charge is placed; call these charge q_a and q_b.
(a) Find the surface charges, σ_a, σ_b and σ_R on the surface of the cavity of radius "a", radius "b" and on the exterior surface of the conducting sphere respectively.

(b) What is the electric field, $\mathbf{E}(\mathbf{r})$, outside the conductor?

(c) Which of your answers to parts (a) and (b) would change if a third charge, q_c, were brought near (but not inside) the conductor?



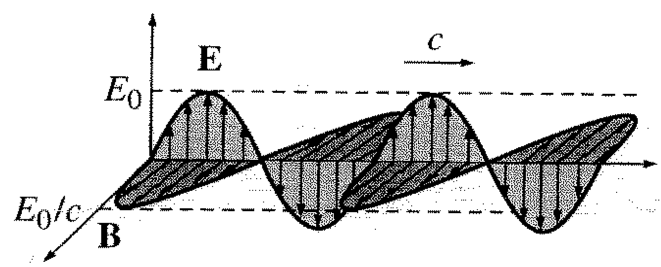
S3: A square hoop, side length "a", with resistance "R", lies a distance "s" from an infinite, straight wire that carries a current, I [see figure]. What is the mutual inductance of the infinite wire with respect to the square hoop?



S4: The figure [below] is a *sketch* of the electric, **E** and magnetic, **B** fields of an electromagnetic (EM) wave.
(a) What features of EM waves are depicted in the sketch?

(b) Write a mathematical form for the **E**(**r**,t) and **B**(**r**,t) fields consistent with this sketch. IF necessary, be clear to say something about your definition of t=0.

(c) Based on your answer to part (b), please label the axes in the figure [below].



S5: Suppose the scaler potential $V = 0$ and the vector potential $\mathbf{A} = A_0 sin(kx - \omega t)\mathbf{y}$, where A_0, ω, and k are constants and "**y**" is a unit vector in the y-direction.
(a) Find the electric, **E**(**r**,t), and magnetic, **B**(**r**,t). fields.

(b) Show that they satisfy **at least 2** of Maxwell's equations in **vacuum**.

Long problems:

L1: Use Maxwell's equations "in matter", see information pages attached to this exam, to derive the general boundary conditions at a **dielectric interface for linear media**:

$$\epsilon_1 E_{1\perp} - \epsilon_2 E_{2\perp} = \sigma_f$$

$$B_{1\perp} - B_{2\perp} = 0$$

$$\mathbf{E}_{1\|} - \mathbf{E}_{2\|} = 0$$

$$(\frac{1}{\mu_1})\mathbf{B}_{1\|} - (\frac{1}{\mu_2})\mathbf{B}_{2\|} = \mathbf{K}_f \times \mathbf{n}$$

where we have allowed for free surface charge, σ_f, and/or free surface currents, $\mathbf{K}_f$, at the interface and "**n**" is a unit vector normal to the interface. **Take care to define clearly what you mean by $\perp$, $\|$, 1, 2 and "n"**; sketches may be the simplest way to make clear your definitions!

L2: Consider electro-magnetic waves at **non-normal** incidence to a dielectric interface, see sketch below. **Assume** that the polarization of the incident wave is **in** the plane of incidence:

$$\mathbf{E_I}(\mathbf{r},t) \;=\; E_{0_\mathrm{I}}\, e^{i(\mathbf{k_I}\cdot\mathbf{r}-\omega t)}\, (cos\theta_\mathrm{I}\mathbf{x} - sin\theta_\mathrm{I}\mathbf{z})$$

where "**x**" and "**z**" are unit vectors in the "x" and "z" directions (see sketch), and E_{0_I} is a complex constant.

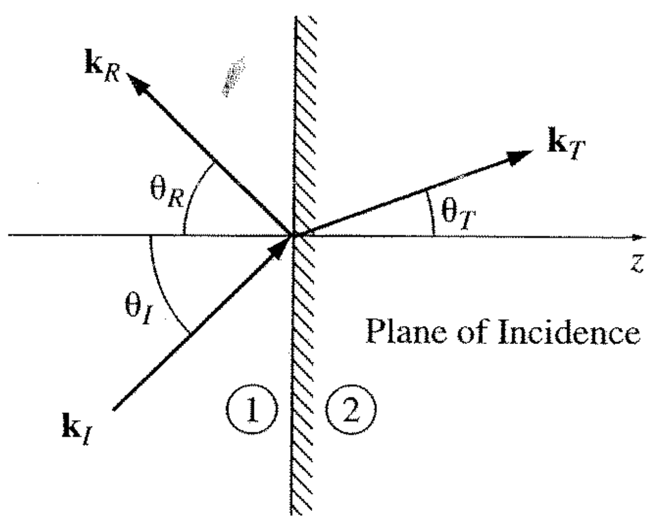
(a) Make a sketch showing **all** of the electric, **E**, and magnetic, **B**, (polarization) vectors.

(b) Just as for $\mathbf{E_I}(\mathbf{r},t)$ above, write expressions for the unknown amplitudes: $\mathbf{B_I}(\mathbf{r},t)$, $\mathbf{E_R}(\mathbf{r},t)$ and $\mathbf{B_R}(\mathbf{r},t)$, $\mathbf{E_T}(\mathbf{r},t)$ and $\mathbf{B_T}(\mathbf{r},t)$ where "I", "R", and "T" denote the incident, reflected and transmitted waves respectively.

Assume that you have already proven Snell's Law and that $\theta_\mathrm{I} = \theta_\mathrm{R}$.

(c) The boundary conditions at the interface of two, linear dielectrics are given in problem, **L1**, above. For **this** problem assume there are **no** free charges, $\sigma_f = 0$, and **no** free currents, $\mathbf{K}_f = 0$. Show how the **E** and **B** fields satisfy **each** boundary condition **and** as a result determine the relationship between: E_{0_R}, E_{0_T} and E_{0_I}.

Be clear and show all steps!!



L3: Current I(t) flows around a small, circular loop (as shown in the figure). Your goal is to determine the Poynting vector in the **radiation zone**, *i.e.* $|\mathbf{r}| >> b$ limit where "b" is the radius of the current loop. For **all** parts below take advantage that we are interested **only** for the potentials and the fields in the **radiation zone**!

(a) Assume that the ring is neutral. What is the scaler potential $V_{rad}(\mathbf{r},t)$?

(b) What approximations can be made in evaluating the vector potential:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}(\mathbf{r}', t_{ret})}{|\mathbf{r}-\mathbf{r}'|}\, dl'$$

where t_{ret} is the retarded time: $t_{ret} = t - |\mathbf{r}-\mathbf{r}'|/c$,
and vectors $\mathbf{r}$ and $\mathbf{r}'$ are defined in the sketch below. **Note: assume that the currents are non-relativistic.**

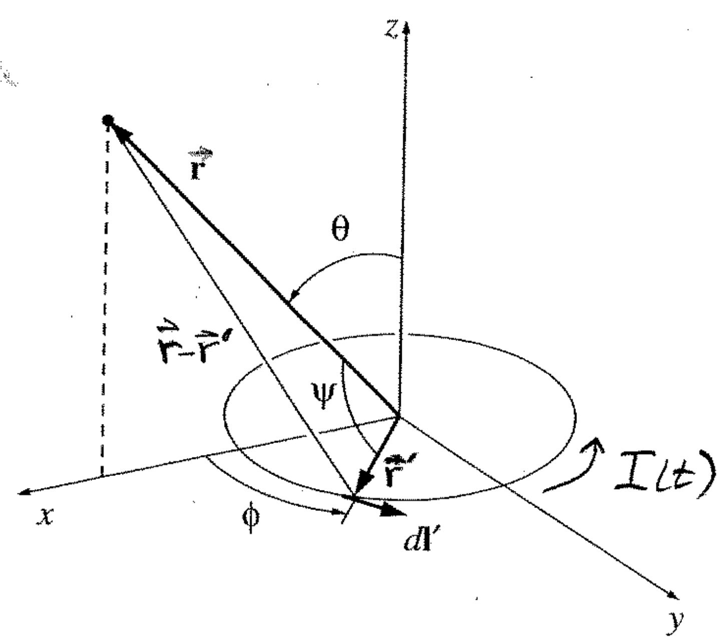
(c) Use the approximations from part (b) to show that:

$$\mathbf{A_{rad}}(\mathbf{r},t) \approx \frac{\mu_0}{4\pi r} \int \dot{\mathbf{I}}(\mathbf{r}', t_0)\, \frac{\mathbf{r}' \cdot \mathbf{r}}{cr}\, dl'$$

where $t_0 = t - r/c$, and "r" is the length of the vector "$\mathbf{r}$": $r = |\mathbf{r}|$.

(d) Use $V_{rad}(\mathbf{r},t)$ and $\mathbf{A}_{rad}(\mathbf{r},t)$ to determine the electric field $\mathbf{E}_{rad}(\mathbf{r},t)$ in the radiation zone.

(e) Finally use $\mathbf{E}_{rad}$ (from part (d)) and $\mathbf{B} = \frac{\mathbf{r}\times\mathbf{E}}{rc}$ to evaluate the Poynting vector in the radiation zone.



FUNDAMENTAL CONSTANTS

ϵ_0	=	$8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	=	$4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	=	$3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	=	$1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	=	$9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \dfrac{1}{2}\displaystyle\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \dfrac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \dfrac{\mu_0}{6\pi c} q^2 a^2$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

VECTOR DERIVATIVES

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\tau = dx\,dy\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient : $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl : $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian : $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\tau = s\,ds\,d\phi\,dz$

Gradient : $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence : $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl : $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian : $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$