Preliminary Examination: Electricity and Magnetism

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Fall 2018

Instructions:

- This exam consists of 10 problems with 10 points each.
- Read all 10 problems before you begin to solve any problem, and solve the problems that seem easiest to you first. Spend your time wisely. If you are stuck on one problem, move on to the next one, and come back to it if you have time after you have solved all other problems.
- Show necessary intermediate steps in each solution. Partial credit will be given if merited.
- No textbook, personal notes or external help may be used other than what is provided by the proctor.
- This exam takes 3 hours.

Potentially Useful Information:

Physical constants and symbols:

ϵ	permittivity,	ϵ_0	vacuum permittivity,
μ	permeability,	μ_0	vacuum permeability,
e	electric charge of the proton,	m_e	mass of the electron,
$c = 1/\sqrt{\epsilon_0\mu_0}$	speed of light,		
ρ	electric charge volume density,	$\mathbf{j}$	electric current density.

Formulas and relations:

- Maxwell's equations:

$$\nabla \cdot \mathbf{D} = \rho_{\text{free}}, \qquad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \qquad \nabla \times \mathbf{H} = \mathbf{j}_{\text{free}} + \partial_t \mathbf{D},$$

  where $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon \mathbf{E}$ and $\mathbf{H} = \mu_0^{-1}\mathbf{B} - \mathbf{M} = \mu^{-1}\mathbf{B}$.

- Lorentz force law:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

- Bio-Savart's law for the magnetic field at position $\mathbf{r}$ due to a steady current element $I\mathrm{d}\boldsymbol{\ell}'$ located at position $\mathbf{r}'$:
$$\mathrm{d}\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{I\mathrm{d}\boldsymbol{\ell}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}.$$
- Poynting vector: $\mathbf{S} = \mathbf{E} \times \mathbf{H}$.
- $Z\mathbf{H} = \hat{\mathbf{k}} \times \mathbf{E}$ and $k = n\omega/c$ for a monochromatic plane wave in medium, where $Z = \sqrt{\mu/\epsilon}$ is the impedance, and $n = \sqrt{\epsilon\mu/\epsilon_0\mu_0}$ is the refractive index.
- The radiation magnetic field at position $\mathbf{r}$ due to a time-varying electric dipole $\mathbf{p}$ at the origin:
$$\mathbf{B}(t,\mathbf{r}) = -\frac{\mu_0}{4\pi r c}\hat{\mathbf{r}} \times \ddot{\mathbf{p}}(t - r/c).$$

Problem 1: Consider four point charges $+q$, $+q$, $-q$ and $-q$ placed on the x-y plane with (x, y) coordinates $(0, a)$, $(0, -a)$, $(-a, 0)$ and $(2a, 0)$, respectively, where $a > 0$. Another point charge Q is placed on the x-axis at a distance $r \gg a$ from the origin. Find the direction and magnitude of the force on Q up to the lowest non-vanishing order in a/r.

Problem 2: The positive terminal of a battery (ground taken at infinity) is attached to a perfectly conducting sphere of radius R in vacuum by a long, thin wire and brings it to potential V. How much work does the battery do in bringing the initially uncharged sphere to potential V? Assume that no energy is lost in radiation or heat.

Problem 3: Two capacitors A and B are identical except that A is filled with a dielectric of relative permittivity $\epsilon_\mathrm{r} = 2$ and B is filled with nothing. Initially A has a charge Q. Find the final charges in A and B when the two capacitors are connected in parallel.

Problem 4: A solid cylinder of radius R and height h is polarized along its axis (chosen to be the z axis) with polarization density $\mathbf{P}$ increasing linearly with height from a value $P_1\hat{\mathbf{z}}$ at the bottom face to $P_2\hat{\mathbf{z}}$ at the top face. Calculate the bound surface charge density and the volume charge density. Show by integrating the total bound charge densities that there is no net charge in the cylinder.

Problem 5: A uniformly charged ring of radius R and charge Q is centered at the origin and rotates in the x-y plane about the z axis with angular velocity ω. Determine the electric and magnetic fields at the center of the ring.

Problem 6: A thin spherical shell of radius R and uniform surface charge density σ rotates at a constant angular speed ω about the z axis which passes through the center of the shell. The magnetic vector potential generated by this spinning shell can be written as

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 R\sigma\omega}{3}\hat{\mathbf{z}} \times \mathbf{r} & \text{if } r \leqslant R, \\ \dfrac{\mu_0 R^4\sigma\omega}{3r^3}\hat{\mathbf{z}} \times \mathbf{r} & \text{if } r \geqslant R, \end{cases}$$

where $\mathbf{r}$ is the position vector. Find out the magnetic field outside and inside the spherical shell.

Problem 7: An infinitely extended plane with a uniform charge density σ coincides with the y-z plane and moves in the z direction with a uniform velocity v. Find the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ at a distance d on either side of the plane.

Problem 8: A point charge q moves with a constant velocity $\mathbf{v}$ in a space filled with constant and uniform electric field $\mathbf{E} = E_0\hat{\mathbf{x}}$ and magnetic field $\mathbf{B} = B_0\hat{\mathbf{y}}$. Find or constrain the three components of the velocity of the point charge (v_x, v_y, v_z).

Problem 9: A monochromatic electromagnetic wave traveling in vacuum is incident normally onto a non-dispersive medium with real relative permittivity $\epsilon_\mathrm{r} = \epsilon/\epsilon_0$ and permeability $\mu_\mathrm{r} = \mu/\mu_0$. By using the boundary conditions dictated by Maxwell's equations, show that a fraction

$$R = \left(\frac{1 - Z_\mathrm{r}}{1 + Z_\mathrm{r}}\right)^2$$

of the incident electromagnetic energy is reflected, where $Z_\mathrm{r} = \sqrt{\mu_\mathrm{r}/\epsilon_\mathrm{r}}$ is the relative wave impedance in the medium.

Problem 10: A point charge q moves along a circular orbit of radius R with angular speed ω. Another point charge $-q$ also moves along the same orbit with the same speed as q but is on the opposite side of the circle. Determine the instantaneous rate of energy loss to the lowest order. What is the condition under which your result is a good approximation of the total radiation power?