Department of Physics and Astronomy, University of New Mexico

E&M Preliminary Examination

Fall 2013

Instructions:

- The exam consists of 10 problems (10 pts each).
- Partial credit will be given if merited.
- Personal notes on the two sides of an 8.5"x 11" sheet are allowed.
- Total time: 3 hours.

Possibly Useful Formulas

- Relation of spherical polar coordinates, (r, θ, ϕ), to Cartesian coordinates:

$$x = r\,\sin\theta\,\cos\phi,\quad y = r\,\sin\theta\,\sin\phi,\quad z = r\,\cos\theta.$$

  Unit vectors:

$$\hat{r} = \sin\theta\,\cos\phi\,\hat{x} + \sin\theta\,\sin\phi\,\hat{y} + \cos\theta\,\hat{z};$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y};\quad \hat{\theta} = \hat{\phi}\times\hat{r}.$$

- Laplacian in spherical polar coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + +\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

- Electric field at position $\vec{r}$ due to a point electric dipole of moment $p\hat{z}$ located at the origin:

$$\vec{E}(\vec{r}) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right),$$

  where $\hat{r}$ and $\hat{\theta}$ are two unit vectors of the spherical polar coordinate system.

- Magnetic field at position $\vec{r}$ due to a point magnetic dipole of moment $\vec{m}$ located at the origin:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3}\right].$$

- Stokes' theorem:

$$\oint_C \vec{A}\cdot d\vec{l} = \iint_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,da,$$

  where S is an open surface bounded by the closed curve C.

- Biot-Savart Law for the magnetic field at position $\vec{r}$ due to a steady current element $Id\vec{\ell}'$ located at position $\vec{r}'$:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{Id\vec{\ell}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}.$$

- Time-averaged power radiated by an oscillating electric dipole:

$$P = \frac{\mu_0|p|^2\omega^4}{12\pi c}.$$

- Time-averaged power radiated by an oscillating magnetic dipole:

$$P = \frac{\mu_0|m|^2\omega^4}{12\pi c^3}.$$

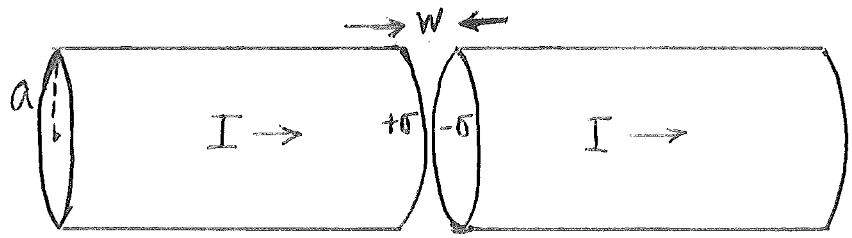
- Instantaneous power radiated by a non-relativistically moving charge with acceleration a (Larmor formula):

- Fresnel formulas for the amplitude reflection coefficient of a plane wave incident at a planar interface between two dielectrics:

$$r_\perp = \frac{n\cos\theta - n'\cos\theta'}{n\cos\theta + n'\cos\theta'};\quad r_\parallel = \frac{n'\cos\theta - n\cos\theta'}{n'\cos\theta + n\cos\theta'},$$

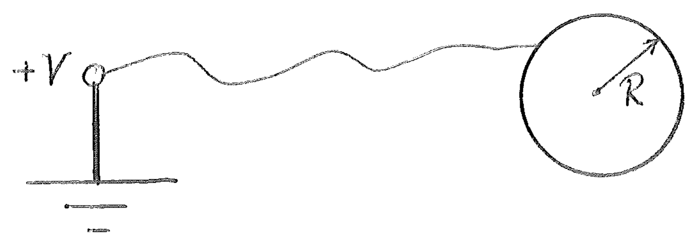
  where $\perp, \parallel$ refer, respectively, to polarizations perpendicular and parallel to the plane of incidence. The angles of incidence and refraction are θ and θ', and n, n' are the refractive indices of the medium of incidence and the medium of transmission, respectively.

1. A capacitor is formed by a small gap w in a wire of radius of radius a (see sketch) where $w << a$ and the initial charge/area σ on either side of the gap is 0. Starting at $t = 0$, a constant current I flows in the wire.



- Derive the electric and magnetic fields as a function of the distance s from the wire axis and time t (for $t \geq 0$) in the gap.
- Find the energy density $u_{EM}(s,t)$ in the electric and magnetic fields, and the Poynting vector $\vec{S}(s,t)$ (magnitude and direction) in the gap.
- Determine the total energy in the gap (*i.e.* out to radius a) as a function of time. Ignore fringing fields.

2. The positive terminal of a battery (ground taken at infinity) is attached to a perfectly conducting sphere of radius R, brining it to potential V. How much work does the battery do in bringing the sphere to the same potential?



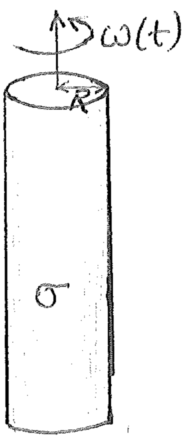
3. An electron with speed v, undergoing cyclotron motion in a transverse magnetic field $B(r)$ at the cyclotron radius r_0, given by $r_0 = mv/[eB(r_0)]$, can be accelerated by ramping the B field in time.

   - Since magnetic fields do no work, what is increasing the kinetic energy of the electron?
   - Show that if the magnetic field at r_0 is half of the average across the orbit,

   $$B(r_0, t) = \frac{1}{2} \frac{\int B(\vec{r}, t)\, da}{\pi r_0^2},$$

   then the radius r_0 of the orbit must be constant in time. Assume nonrelativistic speeds.

4. A charge/area σ is distributed on the surface of a very long cylinder of radius R. The cylinder is spun about its axis so that its instantaneous angular velocity at time t is $\omega(t)$. Find the electric field as a function of position and time.



5. What current density $\vec{J}$ would produce the magnetic vector potential $\vec{A} = ks^2\hat{\phi}$ in cylindrical coordinates, where k is a constant and s is the polar coordinate?

6. A uniformly charged shell of radius R and surface charge density σ is rotating at a constant angular velocity ω about the z axis. At an arbitrary location $\vec{r}$, the magnetic vector potential generated by the spinning sphere is given by the expressions

$$\vec{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R\sigma\omega}{3} r\sin\theta\,\hat{\phi} & \text{for } r \leq R \\ \dfrac{\mu_0 R^4\sigma\omega}{3}\dfrac{\sin\theta}{r^2}\,\hat{\phi} & \text{for } r \geq R. \end{cases}$$

- Show that the magnetic field inside the rotating shell is uniform and along the z direction. What is its magnitude?
- What is the expression for the magnetic field outside the shell? What elementary source can cause such a magnetic field?
- Show that the magnetic field obeys appropriate boundary conditions at the surface of the shell. (*Hint:* Transform to Cartesian coordinates to simplify your calculations.)

7. A monochromatic electromagnetic wave with complex amplitude $\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\exp(-i\omega t)$ travels through a neutral plasma. Show that the current density has the expression

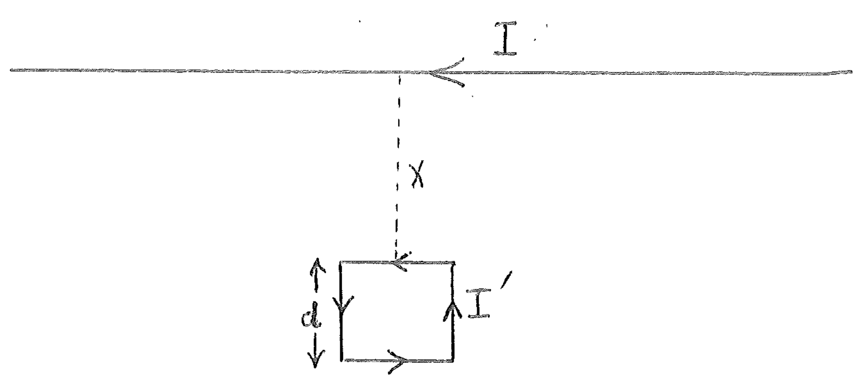
$$\vec{J}(\vec{r},t) = \frac{Ne^2}{-i\omega m}\vec{E}(\vec{r})\exp(-i\omega t).$$

Use Maxwell's equations to show that the electric field inside the plasma obeys the following wave equation:

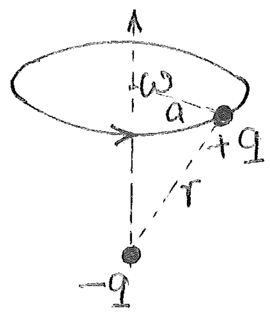
$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{E}(\vec{r}) = \frac{\omega_p^2}{c^2}\vec{E}(\vec{r}),$$

where $\omega_p = \sqrt{Ne^2/(m\epsilon_0)}$ is the plasma frequency.

8. A square loop of wire of side length d is oriented parallel to a long straight wire carrying current I. The loop carries current I' as shown. Determine the force exterted on the loop, assuming that both currents are held constant. Simplify the expression for the force in the limit $x >> d$ and interpret your result.



9. A charge q is set in circular orbit above a charge $-q$ as shown with angular velocity ω. What is the instantaneous rate at which the charge loses energy by electromagnetic radiation? What is the state of polarization of radiation emitted in the vertically upward direction?



10. An infinite line charge, with charge/length λ in the lab frame, moves at a speed v along its length. Using relativistic length contraction, find the electric and magnetic fields (magnitude and direction) in the lab frame and the rest frame of the rod. Verify that the fields in the two frames obey the Lorentz transformation laws for the electromagnetic field,

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \quad \vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B});$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel}, \quad \vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \vec{v} \times \vec{B}/c^2),$$

where c is the speed of light in vacuum and γ is the Lorentz contraction factor.