Department of Physics and Astronomy, University of New Mexico

E&M Preliminary Examination

Fall 2012

Instructions:

- The exam consists of 10 problems (10 pts each).
- Partial credit will be given if merited.
- Personal notes on the two sides of an 8.5"x 11" sheet are allowed.
- Total time: 3 hours.

Possibly Useful Formulas

- Relation of spherical polar coordinates, (r, θ, ϕ), to Cartesian coordinates:

$$x = r\,\sin\theta\,\cos\phi, \quad y = r\,\sin\theta\,\sin\phi, \quad z = r\,\cos\theta.$$

Unit vectors:

$$\hat{r} = \sin\theta\,\cos\phi\,\hat{x} + \sin\theta\,\sin\phi\,\hat{y} + \cos\theta\,\hat{z};$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}; \quad \hat{\theta} = \hat{\phi}\times\hat{r}.$$

- Laplacian in spherical polar coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

- Biot-Savart Law for the magnetic field at position $\vec{r}$ due to a steady current element $I\vec{d\ell}'$ located at position $\vec{r}'$:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{I\vec{d\ell}'\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}.$$

- Time-averaged power radiated by an oscillating electric dipole:

$$P = \frac{\mu_0 |p|^2 \omega^4}{12\pi c}.$$

- Time-averaged power radiated by an oscillating magnetic dipole:

$$P = \frac{\mu_0 |m|^2 \omega^4}{12\pi c^3}.$$

- Instantaneous power radiated by a non-relativistically moving charge with acceleration a (Larmor formula):

$$P = \frac{q^2 a^2}{6\pi \epsilon_0 c^3}.$$

- Fresnel formulas for the amplitude reflection coefficient of a plane wave incident at a planar interface between two dielectrics:

$$r_\perp = \frac{n\cos\theta - n'\cos\theta'}{n\cos\theta + n'\cos\theta'}; \quad r_\parallel = \frac{n'\cos\theta - n\cos\theta'}{n'\cos\theta + n\cos\theta'},$$

  where $\perp, \parallel$ refer, respectively, to polarizations perpendicular and parallel to the plane of incidence. The angles of incidence and refraction are θ and θ', and n, n' are the refractive indices of the medium of incidence and the medium of transmission, respectively.

1. In the Cartesian coordinate system (x, y, z), the electrostatic potential has the form $V = a|z|$, where a is a constant. Does the potential obey the Laplace equation? Derive the specific charge distribution that produces such a potential.

2. A uniformly charged ring of radius R, charge Q, and centered at the origin in the xy plane rotates uniformly at an angular velocity ω about its axis. Determine the electric and magnetic fields at the center of the ring.

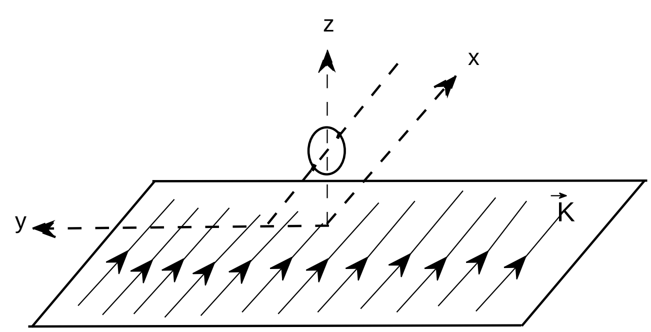
3. A point charge q of mass m is released from rest a distance d from an infinitely extended, uniformly charged plane of surface charge density σ. Take q and σ to have the same sign. Either using the work-energy theorem or otherwise, write down an expression for the acceleration of the charge as a function of its speed without making any non-relativistic approximations. By integrating this expression, obtain the speed of the charge as a function of time. After how long will the charge achieve a speed equal to $0.8c$? Neglect any radiation from the accelerating charge for this problem. *Hint:* You may find useful the indefinite-integral identity,
$$\int \frac{dx}{(1-x^2)^{3/2}} = \frac{x}{(1-x^2)^{1/2}}.$$

4. Consider a charge q of mass m orbiting on a circle under the action of a uniform static magnetic field $\vec{B}$ normal to the plane of the circular orbit. Using the radiative power loss formula for circular orbits,
$$P = \frac{q^2 a^2 \gamma^4}{6\pi\epsilon_0 c^3},$$
where a is the acceleration and γ the relativistic Lorentz factor of the orbiting charge, show that the charge loses energy at a rate proportional to γ^2 as its speed approaches c.

5. A solid sphere of radius a is uniformly polarized with a permanent polarization (density) $\vec{P} = \hat{z}P$ along the z axis. Take the sphere to be centered at the origin. What are the bound volume and surface charge densities in the sphere as a function of the position coordinates (r, θ, ϕ) inside and on the sphere? Show that the potentials
$$V^{<}(r,\theta) = \frac{P}{3\epsilon_0} r\cos\theta, \quad V^{>}(r,\theta) = \frac{Pa^3}{3\epsilon_0 r^2}\cos\theta$$

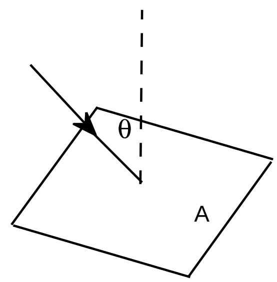
correctly solve the electrostatic problem inside and outside the sphere, respectively, *i.e.,* they both solve the Laplace equation and satisfy the two required boundary conditions at the spherical surface. Using $V^<$, calculate the value of the electric field everywhere inside the sphere.

6. A uniform, infinitely extended current plane of surface current density $\vec{K} = K\hat{x}$ is located in the xy plane. A small circular loop of radius a carrying current I and located above the plane is free to rotate about a diameter that is held fixed and parallel to the x axis.



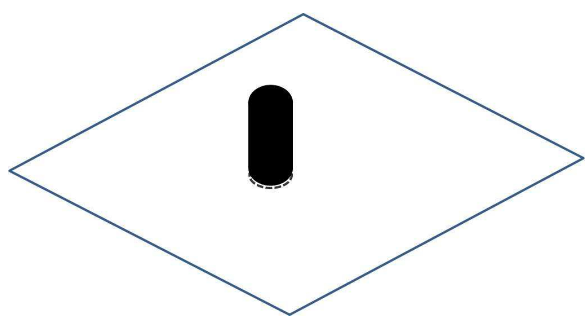
What is the equilibrium orientation of the loop relative to the current plane? At what frequency will the loop perform small oscillations about the equilibrium orientation if it is rotated slightly away from that orientation? Express your answer in terms of μ_0, I, a, and I_m, the moment of inertia of the loop about a diameter.

7. A plane electromagnetic (EM) wave is incident on a large planar metallic foil of area A at angle θ from the normal. The foil is slightly blackened so only a fraction R of the EM energy is reflected and the rest is absorbed.



What are the magnitude and direction of the radiation force generated by the wave in terms of its (time-averaged) intensity I, R, A, and θ?

8. A long skinny bar magnet of magnetization $\vec{M}$ parallel to its axis approaches a highly permeable material with a plane surface. Take the magnetization of the magnet to be perpendicular to the material surface. With what force will the magnet attach to the surface of the material, if the cross-sectional area of the magnet is A?



Make any approximations that are reasonable to arrive at your answers. You may find useful the facts that the magnetic field inside a solenoid of n turns per unit length and current I is $\mu_0 n I$ parallel to the axis of the solenoid and that the force on a uniformly magnetized bar may be written, in perfect analogy with electrostatics, as the effective magnetic charge of amount AM times the external magnetic field $\vec{B}_{ext}$ to which the bar is exposed.

9. An unpolarized monochromatic electromagnetic plane wave of angular frequency ω is incident from vacuum on the plane surface of an ideal plasma. The refractive index of the plasma may be exressed as

$$n(\omega) = \sqrt{1 - \frac{\omega_P^2}{\omega^2}},$$

where ω_P is the plasma frequency which we assume to be a constant and smaller than ω.

   (a) What fraction of the power of the plane wave would be reflected back at normal incidence?

   (b) For what range of values of the angle of incidence would the plane wave be fully reflected?

   (c) For what angle of incidence would the reflected wave be perfectly plane polarized?

10. Consider a monochromatic TEM mode of a planar metallic waveguide of plate separation w that is filled with a dieletric material of refractive index n. If the rms value of the electric field of the mode is E_0 and its angular frequency is ω, then write down complete expressions for the electric and magnetic fields, including their magnitudes and directions, inside the guide. What is the speed of propagation of the TEM mode? Calculate the time-averaged Poynting vector and energy density of the guided mode.