Department of Physics and Astronomy, University of New Mexico

E&M Preliminary Examination

Fall 2010

Instructions:

- The exam consists of 10 problems (10 pts each).
- Partial credit will be given if merited.
- Personal notes on two sides of an 8.5"x 11" sheet are allowed.
- Total time: 3 hours.

Possibly Useful Formulas

- Relation of spherical polar coordinates, (r, θ, ϕ), to Cartesian coordinates:

$$x = r\,\sin\theta\,\cos\phi, \quad y = r\,\sin\theta\,\sin\phi, \quad z = r\,\cos\theta.$$

  Unit vectors:

$$\hat{r} = \sin\theta\,\cos\phi\,\hat{x} + \sin\theta\,\sin\phi\,\hat{y} + \cos\theta\,\hat{z};$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}; \quad \hat{\theta} = \hat{\phi}\times\hat{r}.$$

- Azimuthally symmetric (ϕ-independent) solution of the Laplace equation in spherical polar coordinates:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta),$$

  where the first few Legendre polynomials are defined as

$$P_0(\cos\theta) = 1; \quad P_1(\cos\theta) = \cos\theta; \quad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1); \quad \text{etc.}$$

- Biot-Savart Law for the magnetic field at position $\vec{r}$ due to a steady current element $I\vec{d\ell}$ located at position $\vec{r}'$:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{I\vec{d\ell} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}.$$

- Magnetic field at position $\vec{r}$ due to a point magnetic dipole of moment $\vec{m}$ located at the origin:

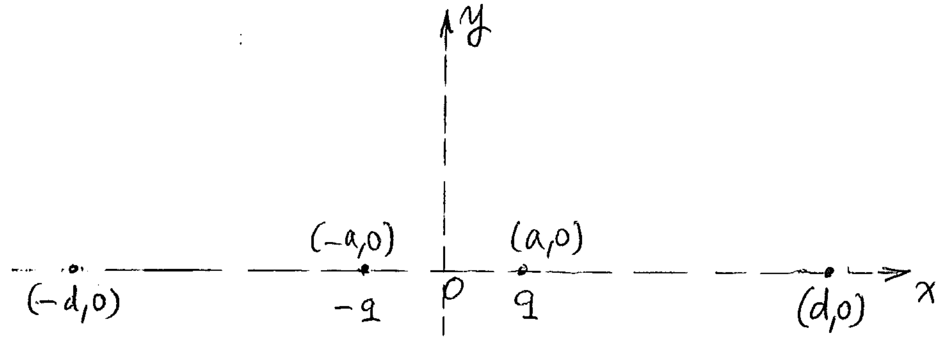
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \left[\frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} \right].$$

- Force on a magnetic dipole $\vec{m}$ in a nonuniform external magnetic field $\vec{B}$:

$$\vec{F} = (\vec{m} \cdot \vec{\nabla})\vec{B}.$$

1. You have been given the electrostatic potential, $V(x, y, z) = a(x^2+y^2-2z^2)$, where a is a positive constant. Show that the potential obeys the Laplace equation. In the Cartesian coordinate system, write down the equations of motion of a charge q of mass m in this potential, and show that the charge cannot be stably bound in this potential.

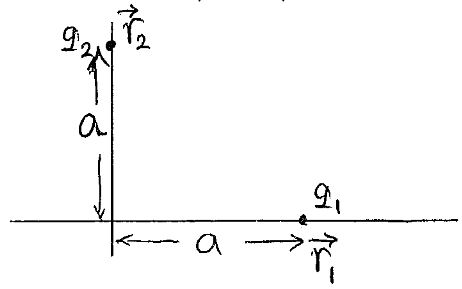
2. Two point charges, $q, -q$, are placed symmetrically about the origin and a distance $2a$ apart along the x axis. What are the potential and electric field at (i) the origin; (ii) a distance $d >> a$ along the positive x axis; and (iii) a distance $d >> a$ along the negative x axis. Your answers in the latter two cases need only be given to the lowest-order non-vanishing terms in powers of a/d.



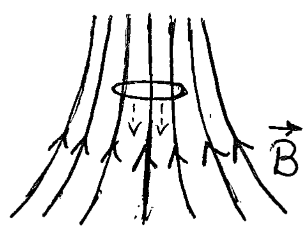
3. Two charges q_1 and q_2 are moving at time t with velocities $\vec{v}_1 = v\hat{x}$ and $\vec{v}_2 = v\hat{y}$ along the x and y directions, respectively. Their position vectors at that time are $\vec{r}_1 = a\hat{x}$ and $\vec{r}_2 = a\hat{y}$. Write down the total force experienced by each charge due to the other. Is Newton's third law obeyed by these forces? Explain your answer. Take the charges to be moving non-relativistically.

   The instantaneous $\vec{B}$ field at position $\vec{r}$ from a non-relativistically moving charge q located at $\vec{r}_0$ and with velocity $\vec{v}$ has the form

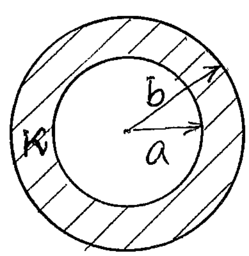
$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times (\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^3}.$$



4. A circular loop of resistance R, mass M, and radius a is dropped from $z = 0$ with its symmetry axis vertical in a magnetic field that is cylindrically symmetric about the vertical (z) axis and whose vertical component is $B_z = Cz$, where C is a positive constant. The axis of the loop falls along the symmetry axis of the magnetic field. In which direction will the induced current flow? Find an expression for the current flowing as a function of the loop velocity. Neglect the contribution to the flux from the field of the induced current.

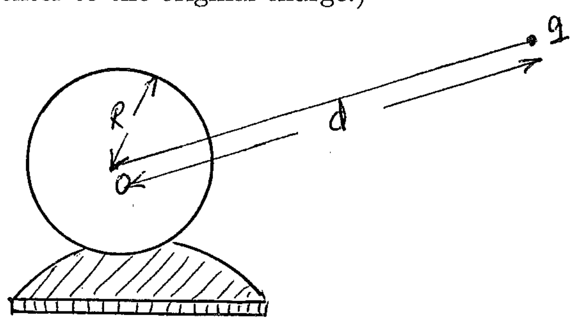


5. A thick, uncharged dielectric spherical shell is placed in a uniform external electric field, $\vec{E}_0$. Let the dielectric constant of the shell be $\kappa = \epsilon/\epsilon_0$. Write down the form of the electric potential everywhere, and argue why the shell behaves as a point eletric dipole for all points outside the shell. Do not solve for the coefficients. What modifications to your answers do you expect in the limit that $\kappa \to \infty$?

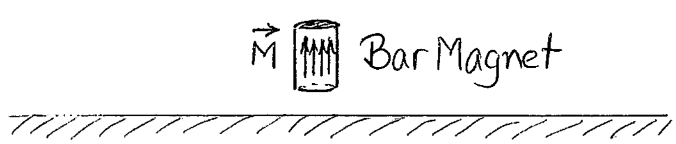


6. A point charge q is placed a distance d from the center of an uncharged perfectly conducting sphere of radius R sitting on a well insulated stand. Take $d > R$. The problem of finding the electric field

outside the sphere may be solved by the method of images. How many image charges and of what amounts and locations are needed within the sphere? What is the electric potential of the sphere? (*Hint:* The image charge corresponding to a real charge q located a distance d from the center of a *grounded* conducting sphere of radius R is of amount $-qR/d$ and is located a distance R^2/d away from the center on the radial line joining the center to the original charge.)

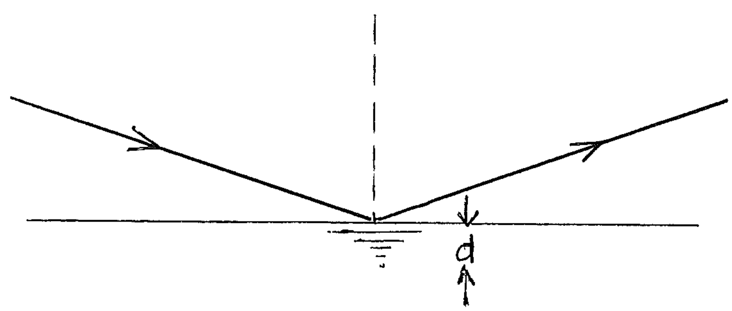


7. A small cylindrical bar magnet uniformly magnetized along its axis is brought close to the planar surface of an otherwise infinitely extended medium. The axis of the bar magnet is maintained normal to the surface of the extended medium at all times. If the medium were a perfect conductor, then would the magnet be (i) repelled, or (ii) attracted, or (iii) unaffected as it approaches the conductor? How would the force scale with distance from the conductor when the distance is large compared to the linear dimensions of the magnet? Repeat for the case the medium is a perfectly permeable medium. (*Hint:* The bar magnet can be described as a collection of co-axial current loops with their axis along the direction of magnetization. The image method may be helpful here.)



8. Two circularly polarized plane waves of opposite helicity but with the same frequency, wave vector, and amplitude are coherently superposed. Let the phase constants of the two waves be ϕ_+ and ϕ_-. Show that the resulting plane wave is linearly polarized. How does the direction of the resulting linear polarization depend on the relative phase, $\Delta\phi \equiv \phi_+ - \phi_-$, between the two initial circular polarizations?

9. A monochromatic plane wave of angular frequency ω propagating inside a medium of refractive index $n = 2$ is incident obliquely on its planar surface with the vacuum. How large must the angle of incidence, θ, be for the wave to be fully reflected by the surface? Let $\sin\theta = 1.5/n$. In terms of the wavelength in vacuum, what is the characteristic distance beyond the surface to which the wave penetrates into the vacuum? (*Hint:* The component of the wave vector parallel to the surface is the same in the medium and the vacuum. Show that this implies a purely imaginary component of the wave vector normal to the surface in the vacuum representing a spatially decaying field in the vacuum away from the surface.)



10. A slab of glass of thickness d and refractive index n is contained between two perfectly conducting planes. Argue qualitatively why the system can serve as a waveguide. What types of waveguide modes are possible in such a system? One way of determining the detailed properties of a mode is to consider the propagation of a plane wave traveling at an angle θ relative to the planes and its successive reflections *ad infinitum* from the bounding planes. Show that when $2knd \sin\theta = 2p\pi$, where p is a positive integer, the reflected waves are in phase upon two successive reflections (*i.e.*, the phase shift in the involved propagation, namely $\overline{\mathrm{AB}} + \overline{\mathrm{BC}}$ in the figure, is a multiple of 2π).