Preliminary Examination: Electricity and Magnetism

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Instructions:

- The exam consists of two parts: 5 short answers (6 points each) and your choice of 2 out of 3 long answer problems (35 points each).
- Where possible show ALL work and partial credit will be given.
- Personal notes on 2 sides of one 8" × 11" page are allowed.
- Total time: 3 hours

Best wishes!!

Short problems:

S1: Derive the capacitance per unit length of a very long cylindrical capacitor with inner (conductor) radius, r_1, and outer (conductor) radius, r_2. Assume the space between r_1 and r_2 is filled with air ... *i.e.* the dielectric constant is essentially 1.0.

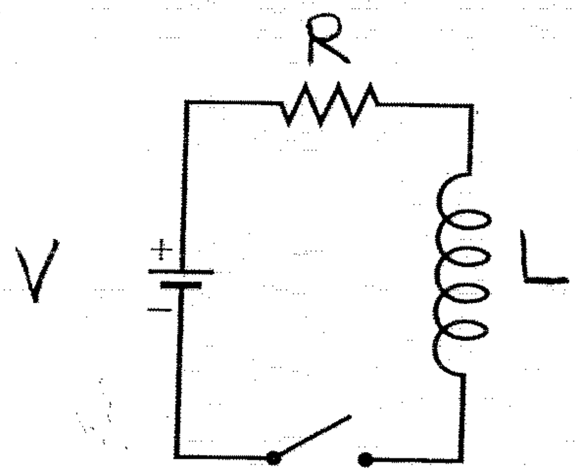
S2: Derive the self-inductance per unit length of a solenoid of radius, a, with n-turns per unit length, and carrying a current, I.

S3: Write down expressions for the (real) electric and magnetic fields for a monochromatic plane wave of amplitude, E_0, and frequency, ω, that is traveling in vacuum in the direction from the origin to the point (1, 1, 1) and with polarization parallel to the xz-plane. Be sure to give the explicit Cartesian components of the propagation (*i.e.* wave) vector, $\mathbf{k}$, and polarization (unit) vector, $\hat{\mathbf{n}}$.

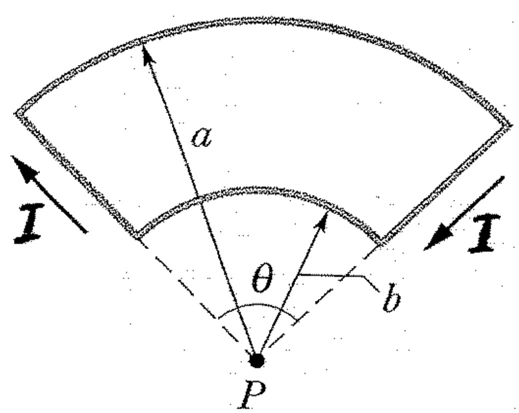
S4: In a simple series "LR" circuit (see sketch), the switch, which is initially open, is closed at time t=0.

(a) What is the current in the circuit as a function of time?

(b) What is the potential difference across the inductor at times: $t = 0$, $t = L/R$, $t = \infty$?



S5: The closed wire loop supports a steady electrical current, I (see sketch). What is the magnetic, **B**, field at point P (which is in the same plane as the loop)?



Long problems:

L1: Picture two tiny metal spheres separated by a distance, d, and connected by a fine wire (see sketch). At time, t, the charge on the upper sphere is $q(t)$ and the charge on the lower sphere is $-q(t)$, and $q(t) = q_0 cos(\omega t)$. In the radiation zone (defined by: $d << c/\omega << r$) the scaler potential, $V(r, \theta, t)$, and vector potential, $\mathbf{A}(r, \theta, t)$, for this oscillating electric dipole, $\mathbf{p}(t) = p_0 \cos(\omega t)\, \hat{\mathbf{z}}$, where $p_0 \equiv q_0 d$, are given by:

$$V(r, \theta, t) = -\frac{p_0 \omega}{4\pi\epsilon_0 c} \left(\frac{cos(\theta)}{r}\right) sin[\omega(t - r/c)]$$

and:

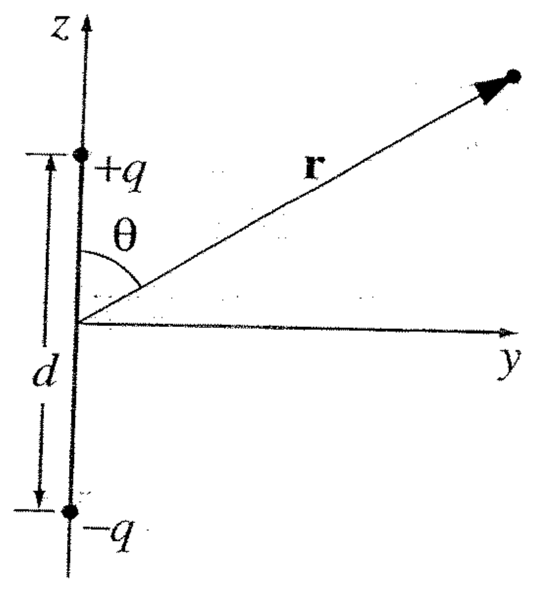
$$\mathbf{A}(r, \theta, t) = -\frac{\mu_0 p_0 \omega}{4\pi r} sin[\omega(t - r/c)]\, \hat{\mathbf{z}}$$

(a) Show that $V(r, \theta, t)$ and $\mathbf{A}(r, \theta, t)$ are consistent with the Lorentz gauge condition.

(b) Evaluate the electric, $\mathbf{E}(r, \theta, t)$, and magnetic, $\mathbf{B}(r, \theta, t)$, radiation fields of the oscillating electric dipole (using $V(r, \theta, t)$ and $\mathbf{A}(r, \theta, t)$ above).

(c) Then show that the time averaged intensity of radiation is given by:

$$< \mathbf{S}(r, \theta) > = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right) \frac{sin^2(\theta)}{r^2}\, \hat{\mathbf{r}}$$

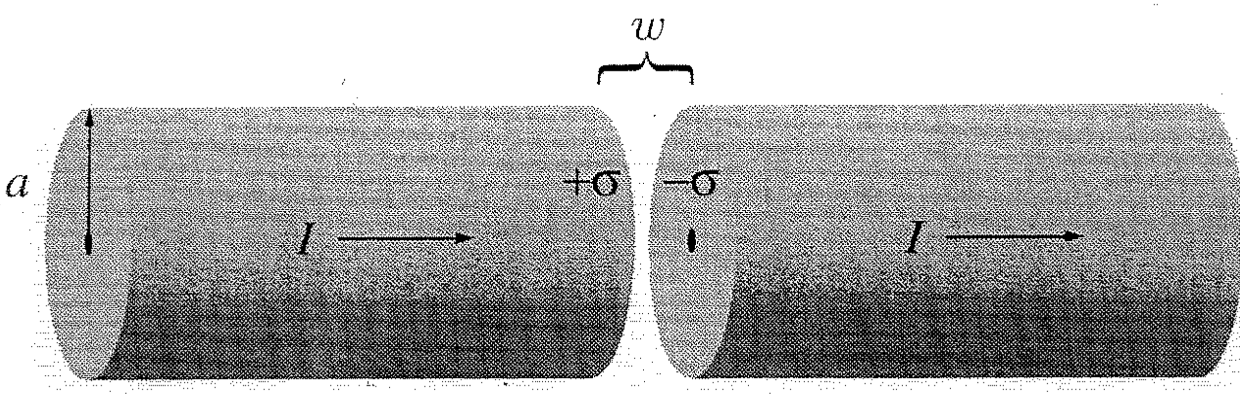


L2: A capacitor is formed by the small gap, w, in a wire of radius, a (see sketch) where $w << a$ and the initial charge/area, σ, on either side of the gap is zero. Starting at t=0, a constant current, I, flows in the wire.

(a) Derive the electric, **E**(s,t), and magnetic, **B**(s,t), fields as a function of distance s, from the wire axis, and time, t (for $t \geq 0$) in the gap.

(b) Find the energy density, $u_{em}(s,t)$, in electric and magnetic fields, and the Poynting vector, **S**(s,t) (magnitude and direction) in the gap.

(c) Determine the total energy in the gap (*i.e.* out to radius $s = a$) as a function of time. Also calculate the total power flowing into the gap. Confirm that the power input equals the rate of increase of energy in the gap with time. (If you are concerned about fringe fields do this calculation for some radius $s = b < a$ well inside the gap).



L3: An infinite straight wire lies a distance, d, above, and parallel to, a grounded conducting plane. The wire is uniformly charged with linear charge density, λ. (For concreteness orient the wire parallel to the x-axis and directly above it and the conducting plane is the xy-plane.)

(a) What is the electric field, $\mathbf{E}(\mathbf{r})$, in the region above the plane?

(b) What is the electric potential, $V(\mathbf{r})$, in the region above the plane?

(c) What is the charge density, $\sigma(x,y)$, induced on the conducting plane?

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

$$\textit{Gradient}: \quad \nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

$$\textit{Divergence}: \quad \nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$\textit{Curl}: \quad \nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

$$\textit{Laplacian}: \quad \nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general :

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy : $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector : $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula : $P = \frac{\mu_0}{6\pi c} q^2 a^2$

FUNDAMENTAL CONSTANTS

ϵ_0	$=$	$8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
μ_0	$=$	$4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
c	$=$	$3.00 \times 10^{8}\,\mathrm{m}/s$	(speed of light)
e	$=$	$1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
m	$=$	$9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$