Department of Physics and Astronomy, University of New Mexico

Classical Mechanics Preliminary Examination

Spring 2019

Instructions:

- This exam consists of 10 problems with 10 points each.
- Read all 10 problems before you begin to solve any problem, and solve the problems that seem easiest to you first. Spend your time wisely. If you are stuck in one problem, move on to the next one, and come back to it if you have time after you have solved all other problems.
- Show necessary intermediate steps. Partial credit will be given if merited.
- No textbook, personal notes, or external help may be used other than what is provided by the proctor.
- This exam takes 3 hours.

Useful Constants, Formulas, and Relations:

- Mass of the Sun: $M_{\rm S} \approx 6 \times 10^{30}$ kg, Radius of the Sun-Earth orbit: $R \approx 1.5 \times 10^{8}$ km
- Moment of inertial of a uniform solid sphere of mass M and radius R: $I = \frac{2}{5}MR^2$
- Moment of inertial of a uniform rod of mass M and length l about its center of mass: $I = \frac{1}{12}Ml^2$
- Moment of inertia of a uniform rectangular plate of mass M, length l, and width w situated in the xy plane about its principal axes:

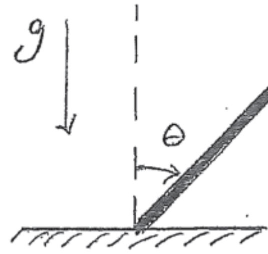
$$\begin{pmatrix} \frac{1}{12}Ml^2 & 0 & 0 \\ 0 & \frac{1}{12}Mw^2 & 0 \\ 0 & 0 & \frac{1}{12}M(l^2+w^2) \end{pmatrix}$$

- Euler's equation for a rotating rigid body:

$$\frac{d\vec{L}}{dt} = \frac{\partial \vec{L}}{\partial t} + \vec{\omega} \times \vec{L}$$

1-A thin pencil with length 15 cm is balanced on a desk top, standing on its point in an almost vertical position. Suppose that initially $\theta = 1 \times 10^{-20}$ rad and $\dot{\theta} = 0$ where θ is the angle of inclination with respect to the vertical. If the point is fixed, how many seconds will it take for the pencil to tip over on its side?
Hint: The following integral, accurate for $\theta_0 < 0.01$, may be useful:

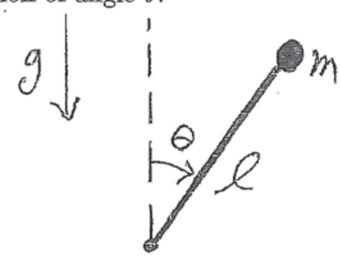
$$\int_{\theta_0}^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta_0 - \cos\theta}} \simeq -\sqrt{2}\ln\theta_0 + 1.695\,.$$



2-A toy rocket has an engine that loses mass at a constant rate k and with a constant relative velocity u. The initial mass of the rocket is m_0 and, since it is a toy, it doesnt go very high so the acceleration due to gravity remains constant. Find the minimum value of u for the rocket to take off when fired, and calculate its velocity as a function of time.

3-Consider a spherically symmetric distirbution of matter with density $\rho(r)$. The circular orbits about the center of distribution have a velcocity $v(r) \propto r^{1/2}$. How does the density $\rho(r)$ varies as a function of the radius?

4-A particle of mass m is attached to a rigid rod of length l and rotates in a vertical plane with constant angular speed ω. Find the magnitude of the force that must be applied to the mass by the rod as a function of angle θ.



5- A ball of mass m slides frictionlessly on ice with constant velocity v. It suddenly hits a patch of the ground with coefficient of kinetic friction μ_k. After some time, it begins rolling without slipping. Find the linear speed of the ball (in terms of v) when it starts to roll without slipping. What fraction of the ball's initial kinetic energy is dissipated?

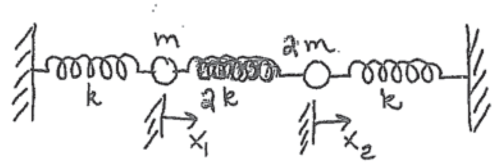
6-A satllite with mass m orbis the Earth with mass M_{E}. The Earth-satellite distance r, and the radial velocity $\dot{r}$ are related to each other according to:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{GmM_{\mathrm{E}}}{r},$$

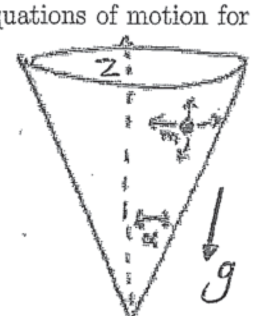
where E is the energy and l is the angular momentum of the Earth-satellite system. Find the relation between E and l so that the aphelion ($r_{\max}$) is four times larger than the prehelion ($r_{\min}$).

7-A homogenous rectangular plate lies in the $x-y$ plane. The principal axes are situated along the x and y directions with $I_{xx} = 2I_0$ and $I_{yy} = 3I_0$. Write the plate's moment of inertia matrix. The plate starts to rotate with the angular velocity $\vec{\omega} = (a, b, 0)$. Find the torque that must be supplied at the initial time for the angular velocity to be constant.

8-Two masses m and $2m$ are attached to each other by a massless spring with spring constant $2k$ and to rigid supports by spring constants k as shown in the figure. Using the displacements x_1 and x_2 from the equilibrium positions of the masses, find the normal frequencies of this system.



9-A particle with mass m moves without friction on the surface of a circular cone with its axis on the vertical z axis and half-angle α. Write down a Lagrangian in terms of the spherical coordinates. Find the equations of motion for r and ϕ, and identify a constant of motion.



10-The solar radiant energy reaches the Earth at an approximate rate of 1.4 kW/m^2. Assume that all this energy results from the conversion of mass to energy. Calculate the rate at which the solar mass is being lost, and use this result to estimate of the lifetime of the Sun in years.