<u>Preliminary Examination: Classical Mechanics</u>

Department of Physics and Astronomy

University of New Mexico

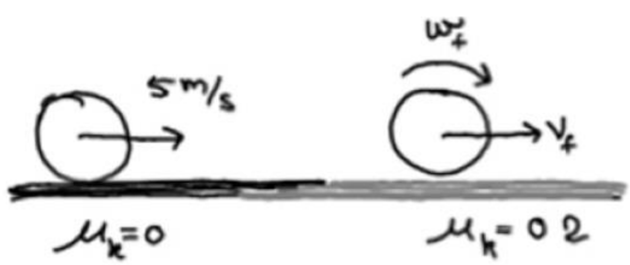
Winter 2016

Instructions:

- The exam consists of 10 problems (10 points each).
- Where possible, show all work; partial credit will be given if merited.
- Useful formulae are provided below; crib sheets are not allowed.
- Total time: 3 hours

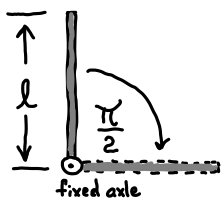
1. A spacecraft with initial mass m_0 is coasting with initial speed v_0 when it enters a very dilute stationary dust cloud with uniform mass density ρ. Assume that the collision cross-sectional area A (for gathering dust) remains constant, and that all dust encountered sticks to the spacecraft. Find an expression for the spacecraft's mass m as a function of time and show that m will increase with time indefinitely provided $v_0 \neq 0$.

2. A bowling ball (a sphere of uniform density) slides at an initial velocity of 5 m/s in the x-direction across a horizontal surface for which the coefficient of sliding friction $\mu_k = 0$. At some point the surface changes composition and in the transition region the friction coefficient monotonically increases along the x axis from 0 to 0.20. What is the final velocity of the center of mass of the bowling ball once it commences rolling without slipping?



3. Consider a hypothetical mine shaft extending diametrically through the earth, connecting the north pole with the south pole. If a rock were released from rest over the shaft at the north pole, how long (in hours) will it take to reach the south pole? How long will it take to return to the north pole? Assume a spherical earth of uniform density and radius 6.4×10^6 m, and neglect friction.

4. A pendulum consists of a rod of length ℓ and uniform density pivoting freely about at axle at one end. Initially the rod is nearly balanced vertically upright, but eventually it tips over. What is the horizontal force on the axle when the rod has rotated half-way around, at the instant it reaches a horizontal position?

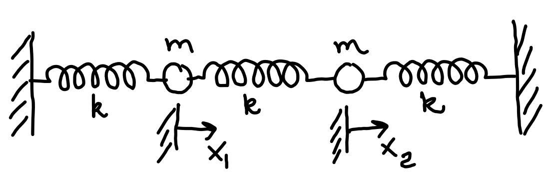


5. Evidence for dark matter comes in part from the observation that the rotation curves showing velocity v as a function of the radius r for visible stars and gas in spiral galaxies such as in our own Milky Way are nearly "flat", independent of r, at odds with what would be expected if the matter were held together by mutual gravitational interactions. It is postulated that instead stars and dust are predominantly held in circular orbits about the galactic center by their attraction to a spherically-symmetric dark matter halo with mass density $\rho(r)$. Determine how ρ must vary with r to account for flat rotation curves.

6. Consider a point-mass hanging at the end of a cord with length ℓ. The mass is set in motion so that it moves in a circle having radius r in the horizontal plane. Find the period of the circular orbit. Neglect the mass of the cord.

7. A proton (mass 0.938 $\mathrm{GeV/c^2}$) with kinetic energy 5 keV scatters from distant alpha particle (helium 4 nucleus, mass 3.727 $\mathrm{GeV/c^2}$) that is initially at rest. For a head-on collision, calculate the radius of closest approach.
(Note: $e^2/4\pi\varepsilon_0 a = 14.4\,\mathrm{keV}$ for $a = 100\,\mathrm{fm}$; 1 eV = 1.6×10^{-19} J)

8. Consider the displacements x_1 and x_2 from equilibrium of two identical masses m connected to each other and to rigid supports by springs with stiffness constant k, as shown in the figure.



Find the natural frequencies of oscillation and sketch their respective mode shapes.

For problems 9 and 10 below, consider the gravitational attraction of two spherically-symmetric bodies with masses m_1 and m_2 respectively is described by the potential energy $U(r) = -\frac{Gm_1m_2}{r}$, that is inversely proportional to the center-to-center separation r . A Lagrangian describing motion relative to the center of mass is given by

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \ ,$$

where θ is the angle in the plane of the motion and $\mu = \frac{m_1m_2}{m_1+m_2}$ is the reduced mass.

9. Show that the total center-of-mass energy $E = \frac{1}{2}\mu\dot{r}^2 + \tilde{U}(r)$ for the two-body system is given by a sum of the radial kinetic energy and the effective potential energy $\tilde{U}(r) = -\frac{Gm_1m_2}{r} + \frac{\ell^2}{2\mu r^2}$, where ℓ is the magnitude of the angular momentum. Sketch a graph of the effective potential energy and the location of the turning points for different values of E, and comment on the distinction between closed (elliptical) and open (hyperbolic) trajectories.

10. For a given value of ℓ, what is the radius R for a circular orbit? If a small kick is applied to one of the two bodies in the radial direction perturbing the radius of a circular orbit, what will be the frequency for oscillations in the radial direction?