Preliminary Examination: Classical Mechanics

Department of Physics and Astronomy

University of New Mexico

Spring 2012

Instructions:

•The exam consists of 10 short-answer problems (10 points each).
•Where possible, show all work; partial credit will be given if merited.
•Personal notes on two sides of an 8×11 page are allowed.
•Total time: 3 hours.

Unless otherwise noted, commonly used symbols are defined as follows:‘

$$
\begin{aligned}
\dot{x} &: \text{ Rate of change of position } x \text{ with time.} \\
\ddot{x} &: \text{ Rate of change of velocity } \dot{x} \text{ with time.} \\
t &: \text{ Time} \\
x(0) &: \text{ Initial position} \\
\dot{x}(0) &: \text{ Initial velocity} \\
L &: \text{ Lagrangian} \\
r &: \text{ Radius} \\
\phi &: \text{ Polar angle in } xy \text{ plane} \\
\theta &: \text{ Azimuthal angle} \\
\hat{x} &: \text{ Unit vector in the } x \text{ direction.}
\end{aligned}
$$

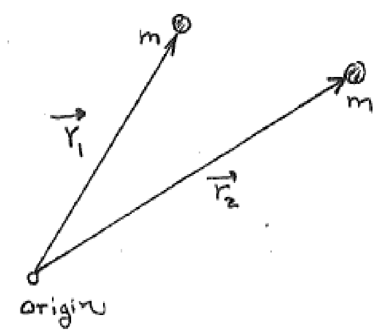
P1: A Lagrangian for a system consisting of two electrons, each with mass m, is given by

$$L = m\left|\frac{d}{dt}\vec{R}\right|^2 + \frac{m}{4}\left|\frac{d}{dt}\vec{r}\right|^2 - \frac{e^2}{4\pi\epsilon_0\left|\vec{r}\right|}$$

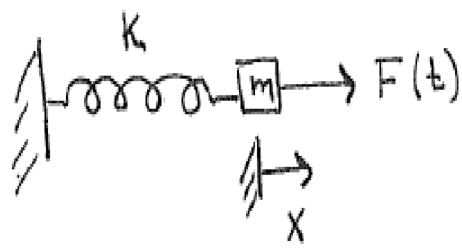
where $\vec{R} = \frac{1}{2}\left(\vec{r}_1 + \vec{r}_2\right)$ is the location of their center of mass, and $\vec{r} = \vec{r}_1 - \vec{r}_2$ is their relative displacement. Expand $\vec{r}$ in spherical coordinates, and $\vec{R}$ in Cartesian coordinates, and show that

$$L = m\left(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2\right) + \frac{m}{4}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - \frac{e^2}{4\pi\epsilon_0 r}.$$

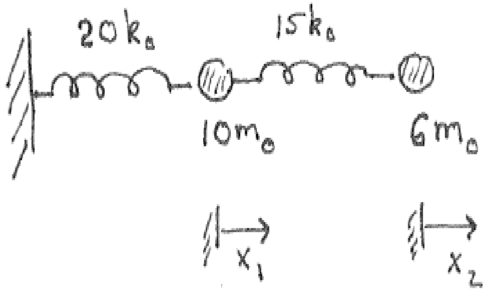
Find up to 6 constants of the motion.



P2: A mass m connected to a rigid wall by a spring with stiffness constant k moves in the x direction in a viscous medium having a linear (Stokes) damping coefficient b. It is subjected to a periodic force $F(t) = F_0 \cos(\Omega t)$ in the x-direction. Derive an expression for the position $x(t)$ of the mass in steady state, and find the amplitude when the oscillator is driven at its resonant frequency.



P3: Consider two masses $m_1 = 10m_0$ and $m_2 = 6m_0$ coupled to each other and to a rigid wall by two springs with stiffness constants $20k_0$ and $15k_0$, as shown in the figure. Find the frequencies of the normal modes, expressing them as multiples of $\omega_0 = \sqrt{k_0/m_0}$.



P4: A pendulum consists of a mass m that swings from the ceiling on the end of a stretchy piece of chewing gum. Initially the length of the gum is ℓ_0, but this increases linearly with t, under the weight of the mass, such that $\ell(t) = \ell_0 + mgt/b$, where b is the gum's viscoelastic damping coefficient. The displacement of the pendulum from the vertical is measured by the angle θ, as shown in the figure. Show that, for small angles, the evolution of θ is governed by the second order differential equation,

$$\ddot{\theta} + \frac{2mg}{b\ell_0 + mgt}\dot{\theta} + \frac{gb}{b\ell_0 + mgt}\theta = 0.$$

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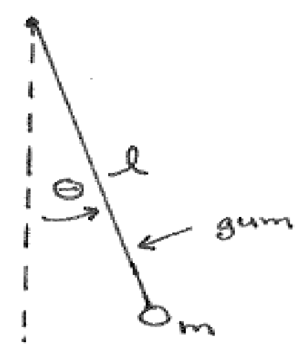
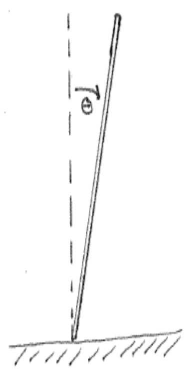


P5. A thin pencil with length 20 cm is balanced on a desk top, standing on its point so that its angle of inclination θ with respect to the vertical is nearly zero. A small perturbation is sufficient to tip it over. Suppose that initially, $\theta(0) = 1 \times 10^{-17}$ radians and $\dot{\theta}(0) = 0$. Assuming that the pencil point remains fixed, in how many seconds will it tip over on its side, through an angle of 90 degrees?

Hint: The following integral, accurate for $\theta_0 < 0.01$, may be useful:

$$\int_{\theta_0}^{\pi/2} \frac{d\theta}{\sqrt{\cos\theta_0 - \cos\theta}} \simeq -\sqrt{2}\ln\theta_0 + 1.695$$



$I_{rod} = \frac{1}{12} m l^2$ (about center of mass)

P6. A particle with mass m and charge q moves in a uniform magnetic field $\vec{B} = B\hat{z}$. Write a Lagrangian describing the motion of the particle in the xy plane that gives the correct Lorentz-force equation of motion,

$$m\,\vec{a} = q\,\vec{v} \times \vec{B}.$$

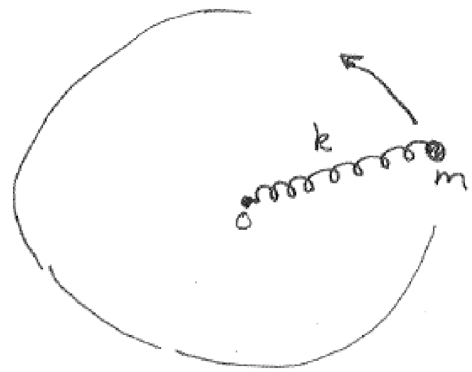
P7. According to specifications, the five Saturn V booster engines collectively supplied a liftoff thrust of 34×10^6 N. The specific impulse* was 2580 N-s/kg. The initial mass of the rocket was 3.0×10^6 kg. Given these specifications, what was the exhaust velocity, and what was the total acceleration, including the acceleration of gravity, felt by the astronauts on liftoff (in g's)?

*The specific impulse is the ratio of the thrust to the rate of consumption of propellant. It is nearly constant.

P8. A mass m moving in a circular orbit about the origin is attracted by a three dimensional harmonic potential,

$$U(r) = \frac{1}{2}kr^2$$

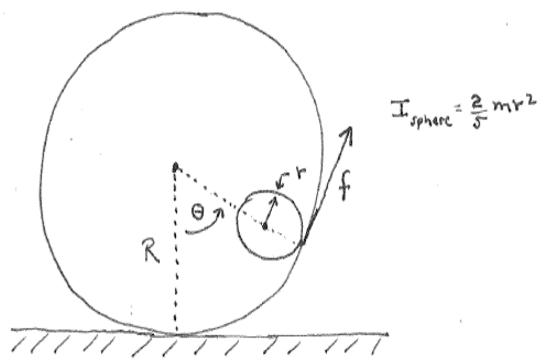
What is the frequency of the orbit? If a small kick is supplied in the radial direction, what will be the frequency of the ensuing small oscillations in r?



P9: A marble with mass m and radius r rolls without slipping on a stationary circular track of radius R fixed in the plane of the vertical, as shown in the figure below. Show that the force of friction f between the marble and the track is given by

$$f = \frac{2}{7} mg \sin\theta,$$

provided that the marble does not lose contact with the track.



P10. A cord of length ℓ and uniform mass density λ is attached to the ceiling by one end, at $x = 0$. The other end passes through a tiny hole in a rigid plate, directly below at $x = \ell$, as shown in the figure. A mass M hangs from the lower end of the cord in order to keep it taut. Write down an expression that describes the tension in the cord as a function of x. Suppose that at a time $t = 0$ the cord is given a small perpendicular displacement $u(x)$ along its length. Show that the linear partial differential equation,

$$\lambda \frac{\partial^2 u}{\partial t^2} = (Mg + \lambda g\,(\ell - x))\,\frac{\partial^2 u}{\partial x^2} - \lambda g \frac{\partial u}{\partial x},$$

describes the subsequent evolution of the transverse wave $u(x, t)$.