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Hydrodynamics at the smallest scales: a solvability criterion for Navier–Stokes equations in high dimensions

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Strong global solvability is difficult to prove for high-dimensional hydrodynamic systems because of the complex interplay between nonlinearity and scale invariance. We define the Ladyzhenskaya–Lions exponent $\alpha(n) = (2 + n)/4$ for Navier–Stokes equations with dissipation $-\Delta^\alpha$ in $\mathbb{R}^n$, for all $n \geq 2$. We review the proof of strong global solvability when $\alpha \geq \alpha(n)$, given smooth initial data. If the corresponding Euler equations for $n > 2$ were to allow uncontrolled growth of the enstrophy $(1/2)\|\nabla u\|_{L^2}^2$, then no globally controlled coercive quantity is currently known to exist that can regularize solutions of the Navier–Stokes equations for $\alpha < \alpha(n)$. The energy is critical under scale transformations only for $\alpha = \alpha(n)$.

Keywords: hydrodynamics; Navier–Stokes; enstrophy

1. Introduction

An important aspect of physical and biological systems relates to fluid flows at meso and nano scales, e.g. fluid flows in nanoporous materials and inside living cells. Hydrodynamic models that neglect fluctuations in the local rate of energy dissipation may not correctly describe mesoscopic and nanoscopic phenomena, especially of non-Newtonian or complex fluids. In the larger context of the importance of small-scale dissipative structures, their formation and collapse, a fundamental question relates to the mathematical properties of physical models of fluid flows.

One of the great unsolved problems in physics and mathematics relates to whether initially smooth solutions of the hydrodynamic equations for three-dimensional fluid flows remain smooth for all time. In two dimensions, the vorticity, defined as a curl of the velocity, is conserved along the flow. Specifically, the vorticity is advected, but cannot be created or destroyed. However, in three

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dimensions, the nonlinearity intrinsic to hydrodynamics allows vorticity to grow (and to decay). The crucial question is whether the vorticity can feed on itself to become ever larger and thereby blow up in finite time. Strong global solvability is precluded by finite-time singularities.

Figure 1 illustrates the basic mechanism. The quadratic nonlinearity in the hydrodynamic equations governing high-dimensional fluid flows allows the creation of Fourier modes corresponding to the sum and difference spatial frequencies of two existing modes. So, it is possible that at some future time, the kinetic energy will have ‘cascaded’ to higher spatial frequencies (i.e. wave numbers). Moreover, these small-scale structures may operate at much faster operational time scales. So it may take less time for the kinetic energy to cascade to the next higher frequencies. The cascade process could thus accelerate as the structures become smaller. In principle, the process can repeat itself ever faster—and endlessly.

Can an initially smooth solution thus become singular in finite time? This crucial question remains open for three-dimensional inviscid flows governed by the Euler equations, as well as for three-dimensional viscous flows of Newtonian fluids. The general topic of blow-ups in nonlinear systems is, in fact, an area of active research [1]. We argue below that the physical origin of the difficulty in reaching an answer involves the interplay between the nonlinear and the scale-invariant aspects of hydrodynamics.

Here, we ask two questions. First, how strong does dissipation have to be in order to avoid finite-time singularities and thereby recover strong global solvability? Second, how does this quantity depend on the number of space

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dimensions? In the following sections, we discuss these issues with the aim of obtaining a deeper insight into fundamental aspects of this important problem. In §2, we specify the system and define the problem. Sections 3 and 4 attack the problem from two different vantage points. We give our concluding remarks in §5.

2. Fractional Navier–Stokes equations

The Navier–Stokes equation for the velocity $u$ of an incompressible fluid with viscous dissipation $-(-\Delta)^\alpha$ is given by

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = -(-\Delta)^\alpha u,$$  \hspace{1cm} (2.1)

where $u$ is a time-dependent divergence-free vector field in $\mathbb{R}^n$ with zero mean. The pressure $p$ can be eliminated using Leray projections, since it merely serves to ensure the divergence-free condition (see equation (4.6) below). Given the initial condition

$$u(x,0) = u_0(x),$$  \hspace{1cm} (2.2)

where $u_0(x) \in C^\infty_c(\mathbb{R}^n)$, the question is whether or not solutions remain smooth.

Ladyzhenskaya [2,3] proved global regularity for the two-dimensional ($n = 2$) parabolic ($\alpha = 1$) case in the early 1960s. For $n = 3$, the challenge is to improve on the well-known exponent $\alpha \geq 5/4$ obtained long ago by Lions [4,5], still the best-known result to date [6–9]. One decade ago, Mattingly & Sinai [8] gave an elementary proof of this result and showed that there are no blow-ups and solutions remain smooth if $\alpha \geq 5/4$. Later, Katz & Pavlović [7] extended the result by proving additional statements, concerning the Hausdorff dimension of the singular set at the time of first blow-up. Recently, in the context of theoretical physics, we have studied [9] the problem in higher dimensions, in order to gain a better understanding of non-Newtonian turbulence and non-local anomalous diffusion of energy density in the Fourier domain in hyper-dissipative incompressible flows.

We draw the reader’s attention to how the lowest value of the exponent $\alpha$ known to ensure global regularity increases by $1/4 = 5/4 - 1$, as the number of dimensions increases from $n = 2$ to $n = 3$. The question that we address is whether or not this $1/4$ increase per dimension is a more general principle with deeper significance. Thus motivated, we define the Ladyzhenskaya–Lions exponent

$$\alpha_{L}(n) = \frac{2 + n}{4}, \hspace{1cm} \forall \hspace{0.2cm} n \geq 2,$$  \hspace{1cm} (2.3)

by linear extrapolation from $n = 2$ and $n = 3$. We then show how to generalize to all $n$ the seminal regularity results of Ladyzhenskaya and Lions, starting with the following theorem:

**Theorem 2.1.** If $\alpha > \alpha_{L}(n)$, then one has strong global solvability for the systems (2.1) and (2.2) in $\mathbb{R}^n$.

In fact, we give two different proofs of this theorem. Although it may have gone somewhat unnoticed, Lions himself arrived at the result 40 years ago [5] (see also the paper by Guermond & Prudhomme [6]). Our own physical argument [9] used $L^1$ estimates in the Fourier domain to bound the rate of growth of
maximal norms of the velocity \( u \) and its spatial derivatives. Since then, we have been able to generalize to arbitrary \( n \) an elegant proof of Katz & Pavlović [7] for the case \( n = 3 \), based on \( L^2 \) estimates. We present first the latter proof because it uses standard methods of functional analysis. We then present the second proof, by reformulating in mathematical terms the physical arguments given in Viswanathan & Viswanathan [9]. This second proof is longer, but which nevertheless sheds further insight on how and why the cascades of energy (and vorticity) continue to bedevil attempts to prove or disprove global regularity for the important and well-known special case \( \alpha = 1 \) and \( n = 3 \). For higher dimensions \( n > 3 \), the nonlinearity can lead to an even more violent cascade of energy, such that it becomes progressively harder to bound the uncontrolled growth of the enstrophy, defined as \( \left( \frac{1}{2} \right) \| \nabla u \|_{L^2}^2 = - \left( \frac{1}{2} \right) \langle u, \Delta u \rangle \).

The two methods of proof, though equivalent, shed light on different aspects of the problem and complement one another. Taken together, they seem to suggest that in higher dimensions \( n \geq 3 \), the Ladyzhenskaya–Lions exponent represents a genuine critical point [9]. Both methods rely ultimately (explicitly or implicitly) on conservation of energy to regularize the solutions. Solutions of the Navier–Stokes equations on a time interval \([0, T]\) satisfy

\[
\| u(\cdot, T) \|_{L^2}^2 = \| u_0 \|_{L^2}^2 - \int_0^T \langle (-\Delta)^{\alpha} u, u \rangle.
\]

The second term on the right is called the dissipation term. The closely related Euler equations for inviscid flow do not have the dissipation term.

We first remark on notation. We will use \( \langle \cdot, \cdot \rangle \) always to denote an \( L^2 \) inner product in space. We follow the notation used by Katz & Pavlović [7] throughout this paper for the expression \( A \lesssim B \), to mean \( A \leq CB \), where \( C \) is some constant. This constant may depend on \( T \), the norms of the initial values, and on an \( \epsilon \), which we keep fixed throughout this paper.


3. Proof based on \( L^2 \) estimates

The following is a generalization to \( \mathbb{R}^n \) of the known proof for \( n = 3 \). We briefly sketch the outline of the argument. By taking suitable inner products, one can bound the rate of growth of \( L^p \) Sobolev norms of \( u \). Then, by invoking a carefully chosen Sobolev inequality, one can re-express these bounds in terms of \( L^2 \) norms, at which point the rest of the proof proceeds conventionally and one recovers global solvability.

The dependence on \( n \) enters the final result solely through the Sobolev inequality. Most importantly, the dependence on \( n \) of \( \alpha_1(n) \) offers a new and different perspective on the well-known explanation of why the parabolic Navier–Stokes equations have global regularity in two but not in three space dimensions. Owing to the close relationship between the Euler and Navier–Stokes equations, the literature does not always make clear the significant differences between them in two dimensions. One explanation argues that for \( n = 2 \), the vorticity is conserved, whereas for \( n = 3 \), vorticity and hence enstrophy can grow. This argument actually misses the point for the Navier–Stokes case, since in two dimensions, the Euler equation itself has smooth solutions, rendering dissipation irrelevant. Plausible explanations have their basis on energy considerations.
For \( n = 2 \), the direct energy cascade—if it counterfactually (i.e. hypothetically) existed in the Euler case—would remain weak enough that dissipation with \( \alpha = 1 \) can control it. But for \( n > 2 \), the energy can cascade more violently, such that dissipation with \( \alpha = 1 \) cannot necessarily control it.

In this context, the following proof provides a clear interpretation, viz. that Sobolev embedding becomes more ‘costly’ for larger \( n \). By costly, we mean that the number of derivatives necessary for the embedding increases with \( n \). Here, energy enters the picture (via the norm \( \| u \|_{L^2}^2 \)), and the role of \( n \) becomes clear.

**Proof of theorem 2.1.** Let \( H^\beta \) and \( W^{\beta, p} \) denote the \( L^2 \) and \( L^p \) Sobolev spaces, respectively, over \( \mathbb{R}^n \) with \( \beta \) derivatives. Taking the inner product by pairing equation (2.1) with \( u \), we get

\[
\frac{1}{2} \frac{d}{dt} \| u \|_{L^2}^2 = - \| (-\Delta)^{\alpha/2} u \|_{L^2}^2 \leq - \| u \|_{H^\alpha}^2 + \| u \|_{L^2}^2. \tag{3.1}
\]

Hence, if the solution \( u \) remains smooth up to time \( T \), we have the estimate

\[
\int_0^T \| u \|_{H^\alpha}^2 \, dt \lesssim (1 + T). \tag{3.2}
\]

We next pair equation (2.1) with \((-\Delta)^{\beta/2} u\) in order to estimate \( \partial(\| u \|_{H^\beta}^2)/\partial t \). We obtain

\[
\frac{1}{2} \frac{d}{dt} \| (-\Delta)^{\beta/2} u \|_{L^2}^2 + \langle u \cdot \nabla u, (-\Delta)^{\beta} u \rangle = - \| (-\Delta)^{(\alpha+\beta)/2} u \|_{L^2}^2. \tag{3.3}
\]

We next estimate the nonlinear term

\[
\langle u \cdot \nabla u, (-\Delta)^{\beta} u \rangle = \langle (-\Delta)^{\beta/2} (u \cdot \nabla u), (-\Delta)^{\beta/2} u \rangle. \tag{3.4}
\]

Recall that \( u \) is divergence free. We can bound the absolute value of the left-hand side of equation (3.4) by

\[
\| u \cdot \nabla u \|_{H^\beta} \| u \|_{H^\beta}, \tag{3.5}
\]

using the Cauchy–Schwarz inequality. Next, we can apply the following well-known generalization (e.g. [10]) of Hölder’s inequality:

\[
\| f_1 f_2 \|_{L^p} \leq \| f_1 \|_{L^{p_1}} \| f_2 \|_{L^{p_2}}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.
\]

So we can now bound equation (3.5) by

\[
\| u \|_{W^{0,p}} \| u \|_{W^{\beta+1,q}} \| u \|_{H^\beta}, \tag{3.6}
\]

or equivalently, following Katz & Pavlović [7], by

\[
\| u \|_{W^{1,p}} \| u \|_{W^{\beta,q}} \| u \|_{H^\beta}, \tag{3.7}
\]

with

\[
\frac{1}{2} = \frac{1}{p} + \frac{1}{q}. \tag{3.8}
\]

Now comes the crucial step, where \( n \) enters the picture. We twice invoke the Sobolev embedding theorem to obtain

\[
\| u \|_{W^{0,p}} \| u \|_{W^{\beta+1,q}} \| u \|_{H^\beta} \lesssim \| u \|_{H^\alpha} \| u \|_{H^{\alpha+\beta}} \| u \|_{H^\beta}, \tag{3.9}
\]

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where, using equation (3.8), the embedding is seen to be completely continuous provided that
\[ 2\alpha - 1 > \frac{n}{2}. \] (3.10)

The embedding ‘costs’ \( \frac{n}{2} \) derivatives, whereas we can ‘spend’ \( 2\alpha - 1 \) derivatives. Remarkably, this is the only step of the argument involving \( n \). We note that the same bound is obtained by using equation (3.7) instead of equation (3.6) in equation (3.9).

The rest of the argument proceeds straightforwardly. From the Cauchy–Schwarz inequality, we get
\[
\|u\|_{H^a} \|u\|_{H^{a+\beta}} \leq \delta \|u\|_{H^{a+\beta}}^2 + \frac{1}{\delta} \|u\|_{H^a}^2 \|u\|_{H^\beta}^2.
\]

Combining this with equation (3.3), we get
\[
\frac{d}{dt} \|u\|_{H^\beta}^2 \lesssim \|u\|_{H^a}^2 \|u\|_{H^\beta}^2 + \|u\|_{L^2}^2.
\]

Together with equation (3.2) and invoking Gronwall’s inequality gives global solvability. The sole sufficient condition (3.10) can be rewritten as
\[ \alpha > \frac{2 + n}{4}. \] (3.11)

4. Proof based on \( L^1 \) (or \( L^\infty \)) estimates

We next give a second proof corresponding to the original argument presented in our earlier study [9]. The latter argument has its basis on the analogy or connection between the Navier–Stokes equation and non-local Fokker–Planck equations for describing anomalous diffusion of ensembles of random walkers, such as found in Lévy flights [11]. Fourier transforming the Euler equation converts the nonlinear term into a non-local one. Taking an inner product of the Fourier-transformed equation with the Fourier transform of \( u \), one obtains an equation that conserves energy, in a manner analogous to how Fokker–Planck equations conserve probability. A well-known fact about fractional or non-local Fokker–Planck equations from the study of Lévy flights concerns the relationship between fat tails in the probability-density function and smoothness of the characteristic functions: the characteristic function may not be smooth for fat-tailed probability-density functions having diverging moments. Dissipation in the Navier–Stokes equations, of course, makes kinetic energy decay and adds an element of exponential damping to the Fourier-transformed equation. We can use this damping to control and bound the growth of suitable norms. For convenience, we chose to study the \( L^1 \) norms involving the Fourier transform of \( u \). The bounds then allow us to obtain \( L^\infty \) estimates for \( u \) and its derivatives. Hence, we actually work in the classical Hölder space rather than in the Sobolev space.

We first briefly sketch the outline of the proof. We take the Fourier transform of the Navier–Stokes equation, and then bound the rate of growth of the Fourier transform \( \hat{u}(k,.) \) of \( u(x,.) \), where \( k \) is the Fourier conjugate of \( x \).
Using this inequality, we then bound the ‘moments’ of the absolute value of \( \tilde{u} \). These moments are defined in a manner analogously to statistical moments of probability-density functions. Conservation of energy plays an explicit and important role in the argument because it leads to estimates of lower order derivatives of \( u \) in terms of higher order derivatives, which leads to the final result.

One aspect of the problem that becomes more clear in this argument concerns the crucial role played by scale invariance. We will show in the concluding remarks that the critical value \( \alpha_c(n) = (2 + n)/4 \) for the marginal case corresponds to the scenario where the dissipation effects and the nonlinear effects (or non-local effects in the Fourier domain) have the same scaling. The number of dimensions \( n \) enters the picture in the step involving conservation of energy. This longer proof provides additional and complementary insight into how and why conservation of energy has different effects for different \( n \). (In fact, both the divergence-free condition of incompressibility and conservation of energy become less relevant as \( n \) tends to infinity.)

In what follows, \( u_i \) denotes the Cartesian component \( i = 1, 2, \ldots, n \) of \( u \) and \( |\mathbf{k}| \) is the length of the \( n \)-dimensional vector \( \mathbf{k} \).

**Proposition 4.1.** The Fourier transform \( \tilde{u}_i(k, \cdot) \) of \( u_i(x, \cdot) \) satisfies

\[
\frac{d}{dt} \| |\mathbf{k}|^m \tilde{u}_i \|_{L^1} \leq \sum_{j}^{n} \sum_{\ell}^{m} \left( \begin{array}{c} m \\ \ell \end{array} \right) \| |\mathbf{k}|^\ell \tilde{u}_j \|_{L^1} \| |\mathbf{k}|^{m-\ell+1} \tilde{u}_i \|_{L^1} \\
- \| |\mathbf{k}|^{2a+m} \tilde{u}_i \|_{L^1} + C_{i,m}(t),
\]

for any integer \( m \geq 0 \), where \( C_{i,m} \) depend only on pressure \( p \).

**Proof of proposition 4.1.** Fourier transforming equation (2.1) to eliminate the space derivatives, we obtain

\[
\frac{\partial}{\partial t} \tilde{u}_i(k, t) + \sum_j^d \tilde{u}_j(k, t) \times (i\mathbf{k}_j \tilde{u}_i(k, t)) = -|\mathbf{k}|^{2a} \tilde{u}_i(k, t) - i\mathbf{k}_i \tilde{p}(k, t),
\]

with the nonlinear term becoming a non-local convolution term in Fourier space. Here, \( i = \sqrt{-1} \), whereas \( i \) denotes a component index. We now note that for any complex function \( g(t) \), if \( dg/dt = A(t) - B(t)g(t) \) and \( B \geq 0 \), then \( (d/dt)|g| \leq |A| - B|g| \). Identifying \( g \) with \( \tilde{u}_i \) and \( B \) with \( |\mathbf{k}|^{2a} \), we get from equation (4.2),

\[
\frac{\partial}{\partial t} |\tilde{u}_i(k, t)| \leq \sum_j \tilde{u}_j(k, t) \times (i\mathbf{k}_j \tilde{u}_i(k, t)) - |\mathbf{k}|^{2a} |\tilde{u}_i(k, t)| + |\mathbf{k}_i \tilde{p}(k, t)|.
\]
Multiplying by $|k|^m$ ($m = 0, 1, 2, \ldots$) and integrating out $k$ over the entire Fourier domain $\mathbb{R}^n$, we get

$$\frac{d}{dt} \int_{\mathbb{R}^n} dk |k|^m |\tilde{u}_i(k, t)| \leq \int_{\mathbb{R}^n} dk |k|^m \sum_j n_j(k, t) \times (k_j \tilde{u}_i(k, t))$$

$$- \int_{\mathbb{R}^n} dk |k|^{2 \alpha + m} |\tilde{u}_i(k, t)|$$

$$+ \int_{\mathbb{R}^n} dk |k|^m (|k_i \tilde{p}(k, t)|). \quad (4.4)$$

Hence,

$$\frac{d}{dt} \| |k|^m \tilde{u}_i \|_{L^1} \leq \sum_j \int dk' \| |k + k'|^m \tilde{u}_j \|_{L^1} |k_j' \tilde{u}_i(k', t)|$$

$$- \| |k|^{2 \alpha + m} \tilde{v}_i \|_{L^1} + C_{i,m}(t)$$

$$\leq \sum_j \int dk' \sum_{m-\ell}^m \left(\frac{m}{\ell}\right) \| |k|^\ell \tilde{v}_j \|_{L^1} \| |k|^{m-\ell} |k_j' \tilde{u}_i(k', t)|$$

$$- \| |k|^{2 \alpha + m} \tilde{v}_i \|_{L^1} + C_{i,m}(t),$$

where $C_{i,m}(t) = \| |k|^{m+1} \tilde{p} \|_{L^1}$. The final integration leads to

$$\frac{d}{dt} \| |k|^m \tilde{u}_i \|_{L^1} \leq \sum_j \sum_{m-\ell}^m \left(\frac{m}{\ell}\right) \| |k|^\ell \tilde{v}_j \|_{L^1} \| |k|^{m-\ell+1} \tilde{u}_i \|_{L^1}$$

$$- \| |k|^{2 \alpha + m} \tilde{v}_i \|_{L^1} + C_{i,m}(t). \quad (4.5)$$

The pressure satisfies an elliptic Poisson equation, which we can invert to obtain

$$p = -\Delta^{-1} \text{Tr}(\nabla u)^2. \quad (4.6)$$

It corresponds to the purely non-divergence-free component of the nonlinearity. Henceforth, we ignore $C_{i,m}$ since pressure plays no relevant role, as discussed earlier in the context of Leray projections.

Proposition 4.1 allows us to control the growth of certain norms. To prove the existence of smooth solutions, we need the term with $2\alpha$ in the exponent to dominate over the sums of product term (i.e. the inertial or nonlinear term). If this condition is satisfied, then the norms remain bounded. We next develop a few useful lemmas, which we prove for completeness.

**Lemma 4.2.** Let $\tilde{f}$ denote the Fourier transform of a function $f : \mathbb{R}^n \to \mathbb{R}^n$. If $f \in C^m$, then $\| |k|^\ell \tilde{f} \|_{L^1} \lesssim \| |k|^m \tilde{f} \|_{L^1}^{(\ell+n/2)/(m+n/2)}$ for $\ell \leq m$.  

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Proof. Scale transformations by a factor \( \lambda \) show that the inner product
\[
\langle f, f \rangle = \langle \tilde{f}(k), \tilde{f}(k) \rangle
\]

\[
= \left( \lambda^{-n/2} \tilde{f} \left( \frac{k}{\lambda} \right), \lambda^{-n/2} \tilde{f} \left( \frac{k}{\lambda} \right) \right),
\]
so that \( \tilde{f}(k) \) and \( \tilde{f}_\lambda(k) \equiv \lambda^{-n/2} f(k/\lambda) \) have the same inner product in \( L^2 = H^0 \).

Now,
\[
\| |k|^\ell \tilde{f}_\lambda \|_{L^1} = \lambda^{\ell+n/2} \| |k|^\ell \tilde{f} \|_{L^1}
\]
and
\[
\| |k|^m \tilde{f}_\lambda \|_{L^1} = \lambda^{m+n/2} \| |k|^m \tilde{f} \|_{L^1}.
\]

So for sufficiently large \( \lambda \), we get \( \| |k|^\ell \tilde{f} \|_{L^1} < \| |k|^m \tilde{f} \|_{L^1} \). Moreover,
\[
\frac{\| |k|^\ell \tilde{f} \|_{L^1}^{(\ell+n/2)}}{\| |k|^m \tilde{f} \|_{L^1}^{(m+n/2)}} = \frac{\| |k|^\ell \tilde{f} \|_{L^1}^{(\ell+n/2)}}{\| |k|^m \tilde{f} \|_{L^1}^{(m+n/2)}},
\]
from which it follows that \( \| |k|^\ell \tilde{f} \|_{L^1} < C \| |k|^m \tilde{f} \|_{L^1}^{(\ell+n/2)/(m+n/2)} \) for some finite \( C \), proving the lemma.

Lemma 4.3. Let \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \) be a multi-index and let \( |\beta| \) denote the sum of the components, according to convention. If \( \| |k|^{|\beta|} \tilde{f}(k) \|_{L^1} < +\infty \), then \( f \in C^{|\beta|} \).

Proof.
\[
(2\pi)^{n/2} |f^{(\beta)}(x)| \leq \int_{-\infty}^{\infty} |k|^{|\beta|} |\tilde{f}(k)| |e^{ikx}| \, dk
\]
and
\[
(2\pi)^{n/2} \| f^{(\beta)} \|_{L^\infty} \leq \| |k|^{|\beta|} \tilde{f}(k) \|_{L^1}.
\]

Hence, \( f \) belongs to the Sobolev space \( W^{|\beta|, \infty} \), and consequently to the classical Hölder space \( C^{|\beta|} \).

Proof of theorem 2.1. We first use lemma 4.2 to bound the norms in the inertial terms,
\[
\| |k|^\ell \tilde{u}_j \|_{L^1} \lesssim \| |k|^{m+2\alpha} \tilde{u}_j \|_{L^1}^{(\ell+n/2)/(2\alpha+m+n/2)}
\]
and
\[
\| |k|^{m-\ell+1} \tilde{u}_i \|_{L^1} \lesssim \| |k|^{m+2\alpha} \tilde{u}_i \|_{L^1}^{(m-\ell+1+n/2)/(2\alpha+m+n/2)}.
\]

From proposition 4.1, we retain control so long as the sum of the exponents on the right-hand sides of equations (4.13) and (4.14) is less than unity. The condition to bound the rate of growth of the relevant \( L^1 \) norms then becomes
\[
\frac{m+1+n}{2\alpha+m+n/2} < 1,
\]
from which we recover the condition (3.11). Lemma 4.3 guarantees global solvability, completing the proof. 

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5. Concluding remarks

Attention is traditionally given in the case \( n = 3 \) to vorticity. In our proofs, the key quantity of interest in the Navier–Stokes system clearly is the energy. As noted by Tao [12,13], the energy is a significant globally controlled coercive quantity that, for \( \alpha = 1 \), is critical for \( n = 2 \), but supercritical for \( n > 2 \). However, we note that for \( \alpha \geq \alpha_{L}(n) \), the energy remains both coercive and either critical (\( \alpha = \alpha_{L}(n) \)) or subcritical (\( \alpha > \alpha_{L}(n) \)). We can state these ideas formally.

**Theorem 5.1.** The kinetic energy \( \frac{1}{2} \| u(., T) \|_{L^2}^2 \) is critical under scale transformations only for \( \alpha = \alpha_{L}(n) \).

**Proof.** If \( u \) is a solution, then

\[
u_{\lambda}(x, t) = \frac{1}{\lambda} u \left( \frac{x}{\lambda^{1/(2\alpha-1)}}, \frac{t}{\lambda^{2\alpha/(2\alpha-1)}} \right)
\]

is also a solution [9]. The energy \( E_{\lambda} \) of the new solution scales to

\[
E_{\lambda} = \lambda^{2 - (n/(2\alpha-1))} E.
\]

We thus see that the value \( \alpha_{L}(n) = (2 + n)/4 \) corresponds to a critical value, for which the energy remains invariant under scale transformations, i.e. \( E_{\lambda} = E \). □

**Remark 5.2.** We briefly comment on the marginal case \( \alpha = \alpha_{L} \). We know from proposition 4.1 and equations (4.13) and (4.14) that the problem for the marginal case arises because the nonlinear and dissipation terms scale identically and thus balance each other, so that it is not immediately clear which term dominates. Any small additional consideration can tip the balance. To address this issue, we recall that

\[
\| u(., T_2) \|_{L^2}^2 < \| u(., T_1) \|_{L^2}^2,
\]

for \( T_2 > T_1 \), i.e. the kinetic energy is a strictly decreasing function of time. This reduction in kinetic energy, insignificant for the non-marginal cases \( \alpha < \alpha_{L}(n) \), can by itself rule out any blow-up for the marginal case, as suggested by the criticality condition in equation (5.2).

We could, alternatively, describe the situation from a different perspective, *viz.* blow-ups are energetically forbidden for \( \alpha > \alpha_{L}(n) \), but not for \( \alpha < \alpha_{L}(n) \). The question is what happens exactly at the critical point. In principle, if the kinetic energy were a constant of the motion, then we would not be able to rule out the possibility that almost all the energy might become concentrated into ever smaller regions of space and at an increasingly faster rate. In this scenario, we would not be able to rule out finite-time singularities. However, kinetic energy is a strictly decreasing function of time, hence in the time that it takes to concentrate the energy into a smaller region, enough energy would have dissipated to prevent the formation of singularities in finite time because not only do dissipation and nonlinearity have the same scaling for \( \alpha = \alpha_{L}(n) \), but in addition, the kinetic energy also decays asymptotically to zero at large times.

Finally, we comment on the hypothetical worst-case scenario where the corresponding Euler equations for inviscid flows lead to blow-ups. No globally controlled coercive critical or subcritical quantity is known to exist for \( \alpha < \alpha_{L} \);
therefore, we are forced to consider the possibility that some initial conditions may very well lead to blow-ups in the Navier–Stokes case for \( \alpha < \alpha_L \) in this hypothetical scenario.

For \( n = 3 \), the Euler equations may allow for the uncontrolled growth of the vorticity \( \omega = \nabla \times u \), owing to the vorticity stretching mechanism [14–16]. The enstrophy
\[
\frac{1}{2} \| \omega \|_{L^2}^2 = \frac{1}{2} \langle \nabla \times u, \omega \rangle = \frac{1}{2} \langle u, \nabla \times \omega \rangle = -\frac{1}{2} \langle u, \Delta u \rangle = \frac{1}{2} \| \nabla u \|_{L^2}^2 \quad (5.4)
\]
can thus conceivably blow up because it also is not conserved,
\[
\frac{1}{2} \frac{d}{dt} \| \omega \|_{L^2}^2 = \langle \omega \cdot \nabla u, \omega \rangle. \quad (5.5)
\]
If it could be shown that a blow-up is possible for the Euler equation, then there may indeed be no way to regularize this worst-case scenario with a dissipation having \( \alpha < \alpha_L(3) \). In this context, we recall the statement of Constantin [15]: ‘it is no exaggeration to say that the Euler equations are the very core of fluid dynamics’.

The same argument can be generalized to \( n > 3 \): if \( \| \nabla u \|_{L^2}^2 \) blows up in the Euler case, no way is (currently) known to regularize solutions via dissipation for \( \alpha < \alpha_L(n) \). Although the question remains open, we speculate that the \( \| \nabla u \|_{L^2}^2 \) can blow up in solutions of the Euler equations in three and higher dimensions. This intuition leads to our final statement.

Conjecture 5.3. The Ladyzhenskaya–Lions exponent \( \alpha_L(n) \) is critical in the sense that it separates two regions. Solutions of the hyper-dissipative Navier–Stokes equations in dimensions \( n \geq 3 \) remain smooth for \( \alpha \geq \alpha_L(n) \), whereas for any \( \alpha < \alpha_L(n) \), finite-time singularities of \( \| \nabla u \|_{L^2}^2 \) are possible.

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References


