KILLING-VECTOR REDUCTIONS FOR COMPLEX-VALUED, TWISTING, TYPE-N VACUUM SOLUTIONS

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1 Simpler Description of the Local Metric

The goal of understanding general classes of solutions of Petrov Type N, with non-zero twist, is one that is still not realized. The use of $\mathfrak{hh}$-spaces to forge a different path toward this goal was developed to a reasonable form in 1992.\(^1\) This work pushes that path a step further, in a direction that has interested the more standard analysis for some time: to look for solutions that admit one Killing and also one homothetic vector, to simplify the task.\(^2\)

A general $\mathfrak{hh}$-space is a complex-valued solution of the Einstein vacuum field equations that admits (at least) one congruence of null strings, i.e., a foliation by completely null, totally geodesic two-dimensional surfaces.\(^3\) Those solutions with algebraically-degenerate, real Petrov type have two distinct such congruences. To describe them we use coordinates $\{p, v, y, u\}$, where $p$ is an affine, null coordinate along one null string, $v$ specifies local wave surfaces, and $y$ and $u$ are transverse coordinates, in those surfaces. The metric is determined by $x$ and $\lambda$, functions of $\{v, y, u\}$, which must satisfy three quasilinear pde's, involving two gauge functions, $\Delta = \Delta(x, y)$ such that $\Delta_1 \neq 0 = \Delta_3$, $\gamma = \gamma(v, u)$ such that $\gamma_1 \neq 0 = \gamma_2$. (1)

The equations may be most easily presented by first introducing a non-holonomic basis for the derivatives in these three variables:

$$\partial_1 \equiv \partial_v, \quad \partial_2 \equiv \partial_y, \quad \partial_3 \equiv \partial_u + a \partial_v, \quad \text{with } a \equiv -x_u/x_v, (2)$$

where $\partial_3$ is actually the derivative with respect to $u$ holding the function $x = x(v, y, u)$ constant, instead of $v$, which is then construed as $v = v(x, y, u)$. (Sometimes $F = F(x, y, u) \equiv v_\epsilon \lambda[v(x, y, u), y, u]$ is also useful.) The twist of the solution is then proportional to $a_2$. The constraining pde’s then have the following form:

$$\lambda_{22} = \Delta \lambda, \quad \lambda_{33} + 2a_1 \lambda_3 + a_31 \lambda = \gamma \lambda,$n\lambda_{23} + a_2 \lambda_3 + a_32 \lambda + \frac{1}{2}a_{322} \lambda = 0. (3)$$

The only known non-trivial solution is that due to Hauser\(^4\)

$$a = y + u, \quad \Delta = 3/(8x), \quad \gamma = 3/(8v), \quad x + v = \frac{1}{2}(y + u)^2,$n\lambda = (y + u)^{3/2}f(t), \quad it + 1 \equiv 4v/(y + u)^2, \quad f \text{ a hypergeometric function}. (4)$$

A null tetrad can be given in terms of these quantities, and the associated non-zero
components of the curvature:
\[ g = \omega^1 \otimes \omega^2 + \omega^2 \otimes \omega^1 + \omega^3 \otimes \omega^4 + \omega^4 \otimes \omega^3, \]  
with \( \omega^1 \equiv pd\mu, \omega^2 \equiv Zd\nu + a_1d\nu, \omega^3 \equiv dv - ad\nu, \omega^4 \equiv dp + Edu - Qu^3, \) 
where \( E = \lambda(\lambda a_3 + 2\lambda a_2), \ Z = p/\lambda^2 + a_2, \ Q = p/\lambda^2 + \lambda^2(\lambda_2/\lambda)\), 
and \( 2R_{1213} = 2\gamma_1/p = C^{(1)}, \ 2R_{2323} = 2(\lambda^2/Z)\Delta_1 = \overline{C}^{(1)}. \)

These constraining pde's are unchanged under any one of the following coordinate transformations.\(^1\) In each of them the function denoted by a capital letter is arbitrary but invertible:

**Transf. I:** \( \{v, y, u\} \rightarrow \{\overline{v}, y, u\}, \) \( F, x, \gamma, \Delta \) as scalars;

**Transf. II:** \( \{x, y, u\} \rightarrow \{v, x, u\}, \) \( x, v, \gamma, \Delta \) as scalars;

**Transf. III:** \( \{v, y, u\} \rightarrow \{v, \overline{y}, u\}, \) \( y = Y(\overline{y}), \) \( x, \gamma \) as scalars, while \( \lambda \) scales as \( \lambda = \sqrt{\gamma_0}, \) and \( \Delta \) has an additional term: \( \Delta = \Delta + \frac{1}{\gamma_0}. \)

**Transf. IV:** \( \{v, y, u\} \rightarrow \{v, y, \overline{u}\}, \) \( u = U(\overline{u}), \) \( x, \Delta \) as scalars, while \( \lambda \) scales as \( \lambda = \sqrt{\gamma_0}, \) and \( \gamma \) has an additional term: \( \gamma = \gamma + \frac{1}{\gamma_0}. \)

We refer to \( \gamma = \gamma(u, v) \) and \( \Delta = \Delta(x, y) \) as gauge functions since transformations I and II would allow them to be replaced by \( v \) and \( x \), respectively. However, we will save that freedom for now.

## 2 Killing’s Equations

We reduce the generality of the pde’s by insisting that the metric allow some symmetries. An arbitrary homothetic vector, \( \overline{V} \), constrains the metric and curvature:

\[ L_\nu g_{\alpha\beta} \equiv V_{(\alpha; \beta)} = 2\chi_0 g_{\alpha\beta}, \quad L_\nu \nabla^\alpha \beta = 0 = L_\nu Q^\alpha \beta. \quad \]

When put together with the pde’s for the metric functions, Eqs.(3), these require any prospective homothetic vector to be determined by two functions, \( K = K(u) \) and \( B = B(v, u) \), as follows:

\[ \overline{V} = +(2\chi_0 - B_v)p\partial_p + \left( \frac{\partial_u + a\partial_v - a_v}{a_v} \right) \partial_y + K\partial_u + B\partial_v, \]

along with various constraints on \( \lambda, a, \Delta, \) and \( \gamma, \) relative to \( K \) and \( B. \) We may however use our coordinate freedom(s) to simplify those equations.

Under Transformation I, \( \nu = \overline{V}(v, u) \implies K = K, \overline{B} = K\overline{V}_v + B\overline{V}_v; \)
under Transformation IV, \( \nu = \overline{U}(u) \implies K = \overline{U}_u K, \overline{B} = \overline{B}. \)

Therefore we may always choose coordinates so that \( B = 0 \) and \( K \) is a constant, say +1, and then ask for the constraints on \( \{\lambda, a, \Delta, \gamma\}. \) We now do this for one true Killing vector, which takes the form \( \partial_u - \partial_y, \) and gives us the (known) result\(^1\) that \( a \) and \( \lambda \) must depend only on \( v \) and \( s \equiv y + u, \) while \( \gamma = \gamma(v) \) and \( \Delta = \Delta(x). \)

(This is the usual transverse Killing vector allowed in this problem.)
A second (homothetic) symmetry vector, \( \tilde{H} \), will have the generic form given in Eq.(7), with its own functions \( B \) and \( K \), constrained by the fact that the commutator of two homothetic vectors is a Killing vector,\(^5\) which requires that
\[
\partial_u B = 0 = \partial^2 u K .
\] (9)
By using the translation and scaling freedom for \( v \) and \( u \) still remaining in transformations IV and I, we acquire the following form for our homothetic vector:
\[
\tilde{H} = (2\chi_0 - \mu_0)p\partial_p + s\partial_s + \mu_0v\partial_v , \quad s \equiv y + u ,
\] (10)
and “scaling” equations for each of our dependent variables:
\[
\tilde{H}(a) = (\mu_0 - 1) a , \quad \tilde{H}(\lambda) = (\chi_0 + 1 - \mu_0) \lambda ;
\]
\[
\tilde{H}(\gamma) = -2 \gamma , \quad \tilde{H}(\Delta) = -2 \Delta .
\] (11)
These allow all the original constraint equations, Eqs.(3), to be rewritten in terms of functions of a single variable, \( t \equiv v/s^{\mu_0} \). Calculations for an optimal presentation for those equations are not yet fully completed, and will be presented elsewhere.

References