PHYS 262: QUANTUM MECHANICS II, CHAPTER 40

PROBABILITY - LIKELIHOOD FOR AN EVENT TO HAPPEN. WE GIVE THE PROBABILITY BY ASSIGNING A NUMBER BETWEEN $\emptyset$ AND 1.

$0 \Rightarrow$ NEVER HAPPENS, $1 \Rightarrow$ CERTAINLY HAPPENS

EXAMPLE: FOR AN EQUALLY-BALANCED FAIR COIN, WHAT IS THE PROBABILITY OF FLIPPING TAILS?

TWO EQUALLY LIKELY OUTCOMES (HEADS AND TAILS) $\Rightarrow$ PROBABILITY $= \frac{1}{2} = .5$

EXAMPLE: WHAT IS PROBABILITY OF ROLLING A 5 ON A SIX-SIDED DICE?

SIX EQUALLY LIKELY OUTCOMES $\Rightarrow$ PROB. $= 1/6 = .167$

- WHAT IS THE PROBABILITY OF ROLLING A 3, 4, OR 5?

3 OUT OF 6 OUTCOMES $\Rightarrow$ PROB. $= 3/6 = 1/2$.

NOTICE THAT $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 3/6$ = PROB. FOR 3 + PROB. FOR 4 + PROB. FOR 5

DICE AND COINS ARE EXAMPLES OF DISCRETE PROBABILITY.

DISCRETE - FINITE NUMBER OF OUTCOMES

CONTINUOUS - INFINITELY MANY OF OUTCOMES.

AN EXAMPLE OF A CONTINUOUS PROBABILITY WOULD BE THE OUTCOMES OF A LENGTH MEASUREMENT (ASSUMING INFINITE PRECISION)

PROBABILITY DISTRIBUTION FUNCTION - LISTS THE PROBABILITY FOR ALL OUTCOMES.

EXAMPLE WHAT IS PROBABILITY DISTRIBUTION FUNCTION FOR ROLLING A SIX-SIDED DICE?

OUTCOME	1	2	3	4	5	6
PROBABILITY	1/6	1/6	1/6	1/6	1/6	1/6

OR $P_i = 1/6 \quad i = 1, 2, 3, \ldots 6$

NORMALIZATION - THE SUM OF THE PROBABILITIES MUST BE 1.

$$\boxed{\sum_i P_i = 1}$$

EXAMPLE - WHAT IS THE PROBABILITY OF ROLLING A 1, 2, 3, 4, 5, OR 6?

$P = 1$ (OF COURSE), BUT ALSO $P = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1.$

THIS PROBABILITY FUNCTION IS NORMALIZED.

AVERAGE - WE CAN HAVE FUNCTIONS OF THE OUTCOMES. THE OUTCOMES ARE USUALLY CALLED RANDOM VARIABLES, e.g., IF X_i = VALUE OF THE ROLLED DICE, THEN X_i IS A RANDOM VARIABLE.

THE AVERAGE FOR A FUNCTION $f(x_i)$ IS:

$$\boxed{\bar{f} = \sum_i P_i f(x_i)}$$

EXAMPLE FRANKIE THE SHARK PAYS THE FOLLOWING FOR THE ROLL OF A DICE.

OUTCOME	1	2	3	4	5	6
PAYS	\$10	0	\$-15	\$-15	0	\$10

($-\$15 \Rightarrow$ YOU PAY HIM \$15).

WHAT IS YOUR AVERAGE EARNINGS?

$$\overline{f} = \sum_i P_i f(x_i) = P_1 f(1) + P_2 f(2) + P_3 f(3) + P_4 f(4) + P_5 f(5) + P_6 f(6)$$

$$= \tfrac{1}{6}(10) + \tfrac{1}{6}(0) + \tfrac{1}{6}(-15) + \tfrac{1}{6}(-15) + \tfrac{1}{6}(0) + \tfrac{1}{6}(10)$$

$$= \tfrac{1}{6}(10 - 15 - 15 + 10) = \tfrac{1}{6}(-10) = -1.67$$ $\rightarrow$ YOU PAY FRANKIE \$1.67 ON AVERAGE.

TWO PARTICULAR AVERAGES ARE GIVEN SPECIAL NAMES –

MEAN – FOR $f(x_i) = x_i$, THE MEAN $\boxed{\overline{x} = \sum_i P_i x_i}$ IS WHAT WE MEAN WHEN WE SAY AVERAGE.

STANDARD DEVIATION FOR $f = (x_i - \overline{x})^2$, THE STANDARD DEVIATION $\Delta x = \sqrt{\overline{f}}$ GIVES THE "SPREAD" OF THE OUTCOMES.

IT CAN BE SHOWN THAT $\boxed{\Delta x = \left[\overline{x^2} - (\overline{x})^2\right]^{1/2}}$

EXAMPLE – WHAT IS THE MEAN AND STANDARD DEVIATION FOR A SIX-SIDED DICE?

$$\overline{x} = \sum_i P_i x_i = \tfrac{1}{6}(1) + \tfrac{1}{6}(2) + \tfrac{1}{6}(3) + \tfrac{1}{6}(4) + \tfrac{1}{6}(5) + \tfrac{1}{6}(6) = \tfrac{1}{6}(1+2+3+4+5+6) = \tfrac{21}{6} = 3.5$$

$$\overline{x^2} = \sum_i P_i x_i^2 = \tfrac{1}{6}(1^2) + \tfrac{1}{6}(2^2) + \tfrac{1}{6}(3^2) + \tfrac{1}{6}(4^2) + \tfrac{1}{6}(5^2) + \tfrac{1}{6}(6^2) = \tfrac{1}{6}(91) = 15.2$$

$$\Rightarrow \Delta x = \left[15.2 - (3.5)^2\right]^{1/2} = 1.7$$

IF FRANKIE THE SHARK GAVE YOU AN AMOUNT OF MONEY EQUAL TO THE OUTCOME (\$1 for a 1, \$2 for a 2, \$3 for a 3, etc.).

ON AVERAGE YOU WOULD WIN \$3.5 PER ROLL. AND YOU WIN SOMEWHERE BETWEEN \$3.5 + \$1.7 = \$5.2 AND \$3.5 − \$1.7 = \$1.8 PER ROLL.

CONTINUOUS PROBABILITY – WITH INFINITELY MANY OUTCOMES, THE PROBABILITY DISTRIBUTION FUNCTION IS A CONTINUOUS FUNCTION.

$$\boxed{P_i \rightarrow P(x)}$$

ALL SUMS ARE REPLACED WITH INTEGRATION.

NORMALIZATION: $\boxed{\int_{-\infty}^{\infty} P(x)\,dx = 1}$

AVERAGE: $\boxed{\overline{f} = \int_{-\infty}^{\infty} P(x) f(x)\,dx}$

MEAN: $\boxed{\bar{x} = \int P(x)\, x\, dx}$ STANDARD DEVIATION: $\boxed{\Delta x = \left[\overline{x^2} - (\bar{x})^2\right]^{1/2}}$ STILL

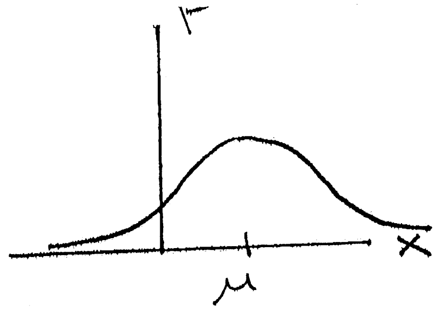
THE PROBABILITY FOR THE OUTCOME TO BE IN THE REGION (a,b) IS

$$\boxed{P(a,b) = \int_a^b P(x)\,dx}$$

NOTICE TECHNICALLY $P(x=a) = \int_a^a P(x)\,dx = 0$. WHICH IS WHY $P(x)$ IS PROBABILITY TO BE INFINITESIMALLY CLOSE TO x.

AN EXAMPLE OF A CONTINUOUS PROBABILITY IS THE BELL CURVE, AKA THE NORMAL DISTRIBUTION.

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



- SHOW THAT $\frac{1}{\sqrt{2\pi}\,\sigma}$ IS A NORMALIZATION FACTOR

$$\int_{-\infty}^{\infty} P(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let $V = \frac{(x-\mu)}{\sqrt{2}\,\sigma} \Rightarrow dv = \frac{dx}{\sqrt{2}\,\sigma} \Rightarrow dx = \sqrt{2}\,\sigma\, dv$

$$\Rightarrow \int_{-\infty}^{\infty} P(x)\,dx = \frac{\sqrt{2}\,\sigma}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-v^2} dv = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} dv$$

$$\boxed{\int_{-\infty}^{\infty} e^{-v^2} dv = \sqrt{\pi}}$$ GAUSSIAN INTEGRAL

$\Rightarrow \int P(x)\,dx = 1$ (AS IT MUST).

- FIND THE MEAN

$$\bar{x} = \int_{-\infty}^{\infty} P(x)\, x\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} x\, dx$$

$V = \frac{(x-\mu)}{\sqrt{2}\,\sigma} \Rightarrow dx = \sqrt{2}\,\sigma\, dv,\ x = \sqrt{2}\,\sigma V + \mu \Rightarrow \bar{x} = \frac{1}{\sqrt{\pi}} \int e^{-v^2} (\sqrt{2}\,\sigma V + \mu)\, dv$

$$= \sqrt{\frac{2}{\pi}}\,\sigma \underbrace{\int e^{-v^2} v\, dv}_{\emptyset} + \frac{\mu}{\sqrt{\pi}} \underbrace{\int e^{-v^2} dv}_{\sqrt{\pi}} \Rightarrow \bar{x} = \mu$$

I LEAVE IT AS AN EXCERCISE TO SHOW $\Delta x = \sigma$