PHYS 262 : QUANTUM MECHANICS, CHAPTER 40

QUANTUM MECHANICS – MOTION OF MATTER WAVES :–

QUANTUM MECHANICS IS THE FUNDAMENTAL THEORY OF MOTION. NEWTON'S LAWS (AND THE SUCH) RE-EMERGE FROM QUANTUM MECHANICS WHEN THE deBROGLIE WAVELENGTH BECOMES MUCH SMALLER THAN THE SCALE OF THE PROBLEM. THIS IS CALLED THE CORRESPONDENCE PRINCIPLE.

COMPLEX NUMBERS – THE MATHEMATICS OF QUANTUM MECHANICS INVOLVES COMPLEX (IMAGINARY) NUMBERS.

$\sqrt{-1} = i$ → ANY NUMBER INCLUDING i IS A COMPLEX NUMBER.

e.g. $3i$, $1+2i$, $7-5i$, e^{5i} → NOTICE THE WAY MOST OF THEM WERE WRITTEN.

ANY COMPLEX NUMBER CAN BE WRITTEN AS $\boxed{z = x+iy}$

x = REAL PART
y = IMAGINARY PART

$$\boxed{Re(z) = x,\ Im(z) = y}$$

COMPLEX CONJUGATE : z^* $\boxed{z^* = x-iy}$ REPLACE "i" WITH "$-i$".

EXAMPLE

z	z^*
$3i$	$-3i$
$1+2i$	$1-2i$
$7-5i$	$7+5i$
e^{5i}	e^{-5i}

ABSOLUTE VALUE HAS A DIFFERENT MEANING FOR COMPLEX #'S

$$|z| = \sqrt{zz^*}$$

$Z = X + iY, \quad Z^* = X - iY \Rightarrow ZZ^* = (X+iY)(X-iY) = X^2 - iXY + iXY - i^2Y^2$

$\Rightarrow ZZ^* = X^2 - i^2Y. \quad i^2 = -1 = (\sqrt{-1})^2 \Rightarrow ZZ^* = X^2 + Y^2 \rightarrow$ ALWAYS REAL AND POSITIVE.

$|Z| = \sqrt{X^2+Y^2}$ IS OFTEN CALLED THE NUMBER'S MAGNITUDE.

EXAMPLE: $|3i|^2 = (3i)(-3i) = 9$

$|1+2i|^2 = 1^2 + 2^2 = 1+4 = 5$

$|e^{5i}|^2 = (e^{5i})(e^{-5i}) = 1$

<u>EULER'S FORMULA</u> – $e^{i\theta}$ PLAYS A VERY IMPORTANT ROLE IN PHYSICS BECAUSE IT COMPACTLY EXPRESSES SINUSOIDAL BEHAVIOR.

TAYLOR SERIES: $\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^7 + \ldots$

$$\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \ldots$$

$$e^X = 1 + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \frac{1}{4!}X^4 + \ldots$$

FIND THE SERIES FOR $e^{i\theta}$ BY LETTING $X = i\theta$.

$$e^{i\theta} = 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \ldots$$

$i^1 = i, \quad i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = i^2 i^2 = (-1)(-1) = 1, \quad i^5 = i^4 i = i, \ldots$

$$\Rightarrow e^{i\theta} = 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{i}{3!}\theta^3 + \frac{1}{4!}\theta^4 + \frac{i}{5!}\theta^5 + \ldots$$

$$= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \ldots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \ldots\right)$$

$$\Rightarrow \boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

EULER'S FORMULA

$\cos\theta = \mathrm{RE}(e^{i\theta})$

$\sin\theta = \mathrm{IM}(e^{i\theta})$

USES FOR EULER'S FORMULA: $\rightarrow$ ANY TRIG IDENTITY YOU NEED.

ANGLE ADDITION: $\cos(\alpha+\beta)$ OR $\sin(\alpha+\beta)$

$$\cos(\alpha+\beta) = \mathrm{Re}\left(e^{i(\alpha+\beta)}\right), \quad \sin(\alpha+\beta) = \mathrm{Im}\left(e^{i(\alpha+\beta)}\right)$$

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$

$$\Rightarrow \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

DOUBLE ANGLE FORMULA: $\cos(2\alpha) = \mathrm{Re}\left(e^{i2\alpha}\right)$, $\sin(2\alpha) = \mathrm{Im}\left(e^{i2\alpha}\right)$

$$e^{i2\alpha} = \left(e^{i\alpha}\right)^2 = (\cos\alpha + i\sin\alpha)^2 = \cos^2\alpha + i^2\sin^2\alpha + 2i\cos\alpha\sin\alpha$$

$$= \cos^2\alpha - \sin^2\alpha + i(2\cos\alpha\sin\alpha)$$

$$\Rightarrow \cos 2\alpha = \cos^2\alpha - \sin^2\alpha, \quad \sin 2\alpha = 2\cos\alpha\sin\alpha$$

TO REVERSE THE EULER EQUATION, WE USE THE FACT THAT

$$e^{-i\theta} = \left(e^{i\theta}\right)^* = \cos\theta - i\sin\theta$$

$$\Rightarrow e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$$

$$e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - \cos\theta + i\sin\theta = 2i\sin\theta$$

$$\Rightarrow \boxed{\begin{aligned} \cos\theta &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\ \sin\theta &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \end{aligned}}$$

WAVEFUNCTION – THE BASIS OF QUANTUM MECHANICS IS THE WAVEFUNCTION: $\psi(x,y,z,t)$. ψ IS RELATED TO THE PROBABILITY OF FINDING A PARTICLE AT THE POINT* (x,y,z) AT THE TIME t.
(* TECHNICALLY THIS IS THE PROBABILITY OF FINDING THE PARTICLE INFINITESIMALLY CLOSE TO (x,y,z).)

WAVEFUNCTIONS EXIST FOR ALL WAVES. FOR AN EM WAVE, THE WAVEFUNCTION IS THE EQUATION FOR THE ELECTRIC OR MAGNETIC FIELD. FOR A PLANE WAVE, $\vec{E} = \hat{\imath} E_0 \cos(kz - \omega t)$ IS ITS WAVEFUNCTION.

THE PHYSICAL INTERPRETATION OF ψ IS MUCH LESS CLEAR. ψ IS A SCALAR AND USUALLY COMPLEX. WHAT IS PHYSICAL IS THE PROBABILITY OF FINDING THE PARTICLE WHICH IS GIVEN BY $|\psi|^2 = \psi\psi^*$.

TO MAKE LIFE EASIER, WE'LL START WITH 1D PROBLEMS $\Rightarrow \psi = \psi(x,t)$.

WAVE EQUATION – ALL WAVEFUNCTIONS OBEY A WAVE EQUATION. FOR EM WAVES, WE HAD $\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$ $\left(\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \text{ IN 1D}\right)$.

THE WAVE EQUATION FOR MATTER WAVES IS THE SCHRÖDINGER EQUATION

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + U\psi = i\hbar\frac{\partial \psi}{\partial t}}$$

U = POTENTIAL ENERGY
NOTICE → SINGLE DERIVATIVE WITH RESPECT TO TIME.

TO MAKE THE SITUATION EVEN EASIER, WE FIND THE STATIONARY STATES.

STATIONARY STATE – ANY SOLUTION TO THE SCHRÖDINGER EQUATION IN WHICH THE PROBABILITY FOR FINDING A PARTICLE IS INDEPENDENT OF TIME.

THIS HAS THE EFFECT OF REQUIRING THE PARTICLE TO HAVE A CONSTANT* ENERGY VALUE OF E.

(* ALSO THE ENERGY IS DEFINITELY KNOWN)

THE WAVEFUNCTION FOR A STATIONARY STATE: $\boxed{\psi(x,t) = \Phi(x)\, e^{-iEt/\hbar}}$

NOTICE $|\psi|^2 = \psi\psi^* = \left(\Phi(x)\, e^{-iEt/\hbar}\right)\left(\Phi^*(x)\, e^{iEt/\hbar}\right) = \Phi(x)\,\Phi^*(x) = |\Phi(x)|^2$

WHICH IS INDEPENDENT OF TIME.

STATIONARY STATES OBEY A SIMPLER WAVE EQUATION

IF $\psi(x,t) = \Phi(x)\, e^{-iEt/\hbar}$ THEN $\frac{\partial\psi}{\partial t} = \Phi(x)\left(-\frac{iE}{\hbar}\right) e^{-iEt/\hbar}$

$\Rightarrow i\hbar \frac{\partial \psi}{\partial t} = \Phi(x)\, E\, e^{-iEt/\hbar}$

$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \Phi(x)}{\partial x^2}\, e^{-iEt/\hbar}$

SO SCHRÖDINGER EQN BECOMES: $-\frac{\hbar^2}{2M}\frac{\partial^2\Phi}{\partial x^2}\, e^{-iEt/\hbar} + U\Phi\, e^{-iEt/\hbar} = \Phi E\, e^{-iEt/\hbar}$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2M}\frac{\partial^2 \Phi}{\partial x^2} + U\Phi = E\Phi}$$

1D, TIME INDEPENDENT SCHRÖDINGER EQUATION

LET'S SHOW THAT THE STATIONARY STATE FOR A FREE PARTICLE ($U=0$)

IS $\Phi = Ae^{iKX} + Be^{-iKX}$. A, B = CONSTANTS DETERMINED FROM INITIAL CONDITIONS.

$\hbar K = p \Rightarrow$ THE PARTICLE'S MOMENTUM.

$e^{iKX} = \cos KX + i\sin KX$ IS THE EQUATION OF A PLANE WAVE PROPAGATION IN THE $+X$ DIRECTION (e^{-iKX} = NEGATIVE $-X$ DIRECTION)

TIME INDEPENDENT WITH $U=0 \rightarrow \frac{-\hbar^2}{2M}\frac{\partial^2\Phi}{\partial X^2} = E\Phi$

$$\frac{\partial\Phi}{\partial X} = AiKe^{iKX} + B(-iK)e^{-iKX} = iK(Ae^{iKX} - Be^{-iKX})$$

$$\frac{\partial^2\Phi}{\partial X^2} = iK(AiKe^{iKX} - B(-iK)e^{-iKX}) = iK(AiKe^{iKX} + BiKe^{-iKX})$$

$$= iK(iK)(Ae^{iKX} + Be^{-iKX}) = i^2K^2\Phi = -K^2\Phi$$

$$\Rightarrow \frac{-\hbar^2}{2M}(-K^2\Phi) = E\Phi \Rightarrow \frac{\hbar^2K^2}{2M} = E \rightarrow \text{SO IF } \frac{(\hbar K)^2}{2M} = E$$

THIS EQUATION SATISFIES SCHRÖ EQUATION.

A FREE PARTICLE ONLY HAS KINETIC ENERGY $\Rightarrow E = \frac{1}{2}MV^2$

$$p = MV \Rightarrow V = \frac{p}{M} \Rightarrow E = \frac{M\left(\frac{p}{M}\right)^2}{2} = \frac{Mp^2/M^2}{2} = \frac{p^2}{2M}$$

$E = \frac{p^2}{2M}$ VS. $E = \frac{(\hbar K)^2}{2M} \Rightarrow p = \hbar K$ OR $K = p/\hbar$ IF IT MAKES MORE SENSE TO YOU.