PHYC 303 Quiz 8 Name Solutions

1. A massless spring is hanging vertically, unloaded, from the ceiling. A mass is attached to the bottom and released. How close is the mass to its final resting position after 1 second, given that the final resting position is 2 m below the starting position and that the motion is critically damped?

Harmonic Oscillator

$m\ddot{x}+b\dot{x}+kx=0$

$\ddot{x}+2\beta\dot{x}+\omega_0^2x=0$

Undamped:

$x(t)=A\cos(\omega t-\delta)$

$x(t)=B_1\cos(\omega t)+B_2\sin(\omega t)$

$x(t)=C_1e^{i\omega t}+C_2e^{-i\omega t}$

Overdamped: $x=C_\pm e^{-\beta t\pm\sqrt{\beta^2-\omega_0^2}}$

Critical: $x=C_1e^{-\beta t}+C_2te^{-\beta t}$

Underdamped: $x=Ae^{-\beta t}\cos(\omega_1t-\delta)$

$\omega_1=\sqrt{\omega_0^2-\beta^2}$

Driven:

$\ddot{x}+2\beta\dot{x}+\omega_0^2x=f_0\cos\omega t$

$x(t)=A\cos(\omega t-\delta)+A_{tr}e^{-\beta t}\cos(\omega_1t-\delta_{tr})$

$A^2=\dfrac{f_0^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$

$\delta=\arctan\left(\dfrac{2\beta\omega}{\omega_0^2-\omega^2}\right)$

$Q=\dfrac{\omega_0}{2\beta}=\pi\dfrac{\text{decay time}}{\text{period}}$

Critical damping: $\beta=\omega_0$.

$x=(C_1+C_2t)e^{-\beta t}=2$ @ $t=0$

$\dot{x}=(C_1+C_2t)(-\beta e^{-\beta t})+C_2e^{-\beta t}=0$ @ $t=0$

$C_1=2.\quad C_2=\beta\cdot 2.$

Now $\omega_0=\sqrt{\dfrac{k}{m}}$ At eqm, $mg=k\cdot 2$

so $\sqrt{\dfrac{g}{2}}=\sqrt{\dfrac{k}{m}}=\omega_0=\beta$

$C_1=2\quad C_2=\sqrt{\dfrac{g}{2}}\cdot 2=2\sqrt{5}$

then

$x=(2+2\sqrt{5}t)e^{-\sqrt{5}t}$

at 1s.:

$x=(2+2\sqrt{5})e^{-\sqrt{5}}=0.69$ m.

-OVER-

2. The period of an undamped oscillator increases by a **small** amount ε when damping is added. The initial period is 10 s. What is β, in terms of ε?

Two ways to solve.

① Taylor expand 'til the cows come home!

$$\omega_1 = (\omega_0^2 - \beta^2)^{1/2} \simeq \omega_0\left(1 - \frac{1}{2}\frac{\beta^2}{\omega_0^2}\right)$$

$$T_0 = \frac{2\pi}{\omega_0} \quad T_1 = \frac{2\pi}{\omega_1} = T_0 + \varepsilon$$

$$= \frac{2\pi}{\omega_0\left(1 - \frac{1}{2}\frac{\beta^2}{\omega_0^2}\right)} \simeq \frac{2\pi}{\omega_0}\left(1 + \frac{1}{2}\frac{\beta^2}{\omega_0^2}\right) \quad \text{so} \quad \epsilon = \frac{\cancel{2}\pi}{\omega_0}\cdot\frac{1}{\cancel{2}}\frac{\beta^2}{\omega_0^2}$$

$$\beta^2 = \frac{\epsilon\omega_0^3}{\pi} = \frac{8\pi^2\epsilon}{T_0^3} \qquad \boxed{\beta = \sqrt{\frac{8\pi^2\epsilon}{T_0^3}}}$$

② use fancy-pants calculus!

$$\beta = (\omega_0^2 - \omega_1^2)^{1/2} \qquad \frac{d\beta}{dT} = \frac{d\beta}{d\omega_1}\frac{d\omega_1}{dT} = \frac{1}{\cancel{2}}\beta^{-1}(-\cancel{2}\omega_1)\cdot -1\cdot 2\pi T^{-2} \qquad \left(\omega_1 = \frac{2\pi}{T_1}\right)$$

Problem! This says a small change in T gives an ∞ change in β! (when β=0.) Write it this way:

$$\beta\, d\beta = +2\pi\omega_1 T^{-2}\, dT$$

$$\Downarrow \qquad \downarrow \omega_1 \simeq \omega_0$$

$$\frac{1}{2}\, d(\beta^2) = +2\pi\omega_0 T^{-2}\, dT$$

$d(\beta^2) \Rightarrow \beta^2 \quad dT \Rightarrow \varepsilon$

since β is initially 0.

$$\text{So} \quad \beta^2 = \frac{4\pi\omega_0}{T_0^2}\epsilon = \frac{8\pi^2}{T_0^3}\epsilon.$$