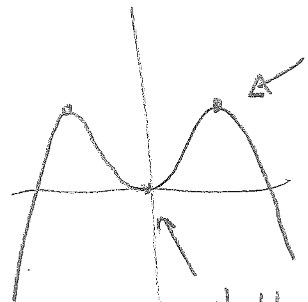
2.52 a) $F = -U' = \frac{U_0}{a}(4x' - 4x'^3) \qquad x' = x/a$

b)



$F = 0 = 4U_0 x'(1 - x'^2)$

$x' = 0, \quad x' = \pm 1.$

$U'' = \frac{4U_0}{a^2}(1 - 3x'^2) \quad \begin{matrix} > 0 & \text{at } x' = 0 \\ < 0 & \text{at } x' = \pm 1. \end{matrix}$

c) Can ignore x'^3 when x' is small

$U'' = \frac{4U_0}{a^2} = k \qquad \omega = \sqrt{\frac{4U_0}{ma^2}}$

Note: if $U = -U_0[2(x/a)^2 - (x/a)^4]$, ± 1 would be stable points

for those, $U'' = \frac{4U_0}{a^2}(3-1) = \frac{12U_0}{a^2}$ so $\omega = \sqrt{\frac{12U_0}{ma^2}}$

d) Needs energy to get over the hill at $x' = \pm 1$.

$U = U_0[2-1] = U_0 = \frac{1}{2}mv^2 \qquad v_{esc} = \sqrt{\frac{2U_0}{m}}$

e) $E = U_0$ so $T = E - U = U_0\left[1 - 2\left(\frac{x}{a}\right)^2 + \left(\frac{x}{a}\right)^4\right] = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$

$\frac{dx}{dt} = \sqrt{\frac{2U_0}{m}[\quad]} \qquad \int \frac{dx}{\sqrt{\frac{2U_0}{m}\left(1-\left(\frac{x}{a}\right)^2\right)^2}} = \int dt$

$\sqrt{\frac{m}{2U_0}} \int \frac{dx}{1-\left(\frac{x}{a}\right)^2} = t = \sqrt{\frac{m}{2U_0}}\left\{\ln\left(\frac{x}{a}+1\right) - \ln\left|\frac{x}{a}-1\right|\right\}\frac{a}{2}$

$$t = a\sqrt{\frac{2m}{u_0}} \ln\left|\frac{1+\frac{x}{a}}{1-\frac{x}{a}}\right|$$

$$\frac{1+\frac{x}{a}}{1-\frac{x}{a}} = \exp\left\{\frac{t}{a}\sqrt{\frac{u_0}{2m}}\right\} \qquad \frac{x}{a} = \frac{e^{t'}-1}{e^{t'}+1} = \frac{e^{t'/2}-e^{-t'/2}}{e^{t'/2}+e^{-t'/2}} = \tanh\left(\frac{t'}{2}\right)$$

$$\equiv t'$$

finally $x = a\tanh\left[\frac{t}{2a}\sqrt{\frac{u_0}{2m}}\right]$ Asymptotically approaches $x = a$ at $t \to \infty$.

2.53 a) $\frac{\partial F_x}{\partial y} = az \quad \frac{\partial F_y}{\partial x} = az$ ✓

$\frac{\partial F_x}{\partial z} = ay \quad \frac{\partial F_z}{\partial x} = ay$ ✓

$\frac{\partial F_y}{\partial z} = ax+b \quad \frac{\partial F_z}{\partial y} = ax+b$ ✓

Conservative $\vec{\nabla}\times\vec{F} = 0$

find $u(x,y,z)$ by $\int$ along x, y, z

$\int F_x dx = ayzx + \frac{bx^2}{2} + cx$

$\int F_y dy = axzy + bzy$

$\int F_z dz = axyz + byz$

omit duplicate terms

$$\boxed{u = -\left[axyz + \frac{bx^2}{2} + cx + byz\right]}$$

(+ const.)

b) $\frac{\partial F_x}{\partial y} = 0 \quad \frac{\partial F_y}{\partial x} = 0$ ✓

$\frac{\partial F_x}{\partial z} = -e^{-x} \quad \frac{\partial F_z}{\partial x} = -e^{-x}$ ✓

$\frac{\partial F_y}{\partial z} = \frac{1}{z} \quad \frac{\partial F_z}{\partial y} = \frac{1}{z}$ ✓

Conservative.

$\int F_x dx = ze^{-x}$

$\int F_y dy = y\ln z$

$\int F_z dz = e^{-x}z + y\ln z.$

$$\boxed{u = -ze^{-x} - y\ln z.}$$

c) $\vec{F} = \frac{a}{r}\hat{r}$. $\vec{\nabla}\times\vec{F} = 0$ since only $\frac{\partial F_r}{\partial r}$ is non-vanishing.

Since $-\vec{\nabla}U = F$ and $\vec{\nabla}U = \hat{r}\frac{\partial U}{\partial r}$, $\frac{\partial U}{\partial r} = -\frac{a}{r}$ $\quad U = -a\ln(r/a)$ + const.

I put an "a" here to make r/a dimensionless.
It can be absorbed into the constant of integration.

4.2. $F = x^2\hat{x} + 2xy\hat{y}$

a) $W = \int_0^1 x^2\,dx + \int_0^1 2\cdot 1\cdot y\,dy = \frac{1}{3} + 1 = 4/3.$

x on top

b) Path $y = x^2$.

$$W = \int F_x dx + F_y dy = \int \left(F_x \frac{dx}{ds} + F_y \frac{dy}{ds}\right) ds$$

Here $s = x$, $y = s^2$,

$\frac{dx}{ds} = 1$ $\quad \frac{dy}{ds} = 2s = 2x$ $\quad$ So $W = \int_0^1 (x^2\cdot 1 + \underbrace{2x(x^2)}_{F_y}\cdot \underbrace{2x}_{\frac{dy}{ds}})\,dx$

$$= \left(\frac{x^3}{3} + \frac{4x^5}{5}\right)\Big|_0^1 = \frac{1}{3} + \frac{4}{5} = \frac{17}{15}$$

c) Path $x = t^3$, $y = t^2$.

$$W = \int_{t=0}^{t=1} \left(F_x \frac{dx}{dt} + F_y\frac{dy}{dt}\right) dt \qquad \frac{dx}{dt} = 3t^2 \quad \frac{dy}{dt} = 2t$$

$$W = \int_0^1 (t^6\cdot 3t^2 + 2t^3t^2\cdot 2t)\,dt = \left(\frac{1}{3}t^9 + \frac{4}{7}t^7\right)\Big|_0^1 = \frac{19}{21}.$$

4.12 a) $f = x^2 + z^3$ $\quad \vec{\nabla} f = 2x\hat{x} + 3z^2\hat{z}$

b) $f = ky$ $\quad \vec{\nabla} f = k\hat{y}$

c) $f = (x^2+y^2+z^2)^{1/2}$ $\quad \frac{\partial f}{\partial x} = \frac{1}{\cancel{2}} \frac{\cancel{2}x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{r}.$ $\quad \vec{\nabla} f = \frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{z}{r}\hat{z}$

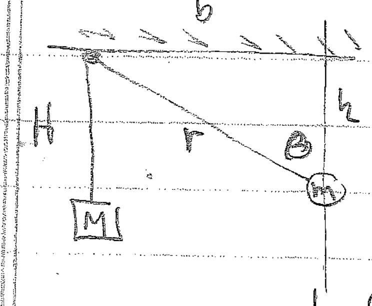
$= \hat{r}$!

d) $f = (x^2+y^2+z^2)^{-1/2}$ $\quad \frac{\partial f}{\partial x} = -\frac{1}{\cancel{2}} \frac{\cancel{2}x}{r^3}$ $\quad \vec{\nabla} f = -\frac{x}{r^3}\hat{x} - \frac{y}{r^3}\hat{y} - \frac{z}{r^3}\hat{z}$

so $\vec{\nabla} f = -\frac{1}{r^2}\hat{r}.$

These last 2 are trivial in spherical coordinates.

4.36



$U = -MgH - mgh$

Need to relate h & H.

$H + (b^2+h^2)^{1/2} = l$ length of string

$U = -Mg\left(l - (b^2+h^2)^{1/2}\right) - mgh$

$\frac{1}{g}\frac{dU}{dh} = M \cdot \frac{1}{\cancel{2}}(b^2+h^2)^{-1/2} \cdot \cancel{2}h) - m = 0$ at eqm.

$\Downarrow$

$\frac{m}{M} = \frac{h}{\sqrt{b^2+h^2}} = \cos\theta.$

$\frac{1}{g}\frac{d^2U}{dh^2} = M\left[\frac{(-\frac{1}{\cancel{2}})h \cdot \cancel{2}h}{(b^2+h^2)^{3/2}} + \frac{1}{(b^2+h^2)^{1/2}}\right]$

$= M\left[\frac{-\cancel{h^2} + (b^2+\cancel{h^2})}{(b^2+h^2)^{3/2}}\right] > 0$ stable