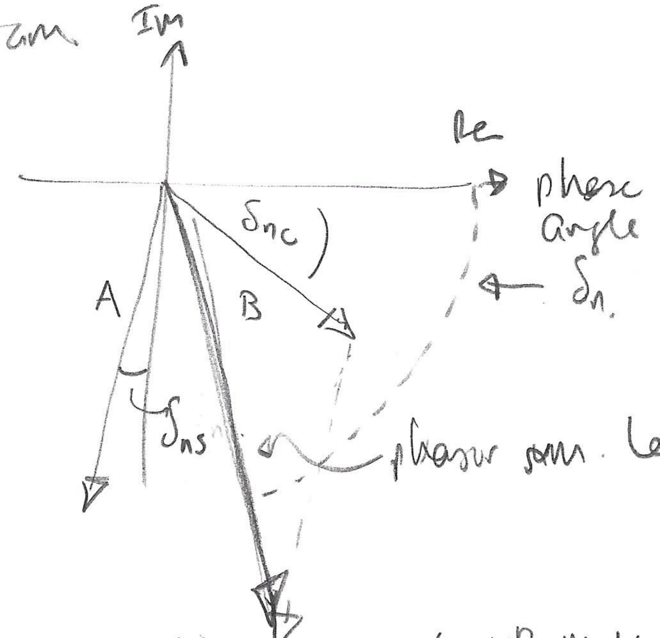
HW 11 Solutions

1. a) $A\sin(n\omega t - \delta_{ns}) + B\cos(n\omega t - \delta_{nc}) = D_n\cos(n\omega t - \delta_n)$.

for any A, B, you can find D_n & δ_n. use a phasor diagram



b) [Since $\cos(0\omega t) = \cos 0 = 1$, we may take $\delta_0 = 0$.]

$$x^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} D_n D_m \cos(n\omega t - \delta_n)\cos(m\omega t - \delta_m)$$

$\cos(n\omega t - \delta_n) = \cos n\omega t\cos\delta_n + \sin n\omega t\sin\delta_n$.

write $c\delta_n = \cos\delta_n$, $s\delta_n = \sin\delta_n$

then
$$x^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} D_n D_m \Big[\cos n\omega t\cos m\omega t\, c\delta_n c\delta_m$$
$$+ \sin n\omega t\sin m\omega t\, s\delta_n s\delta_m$$
$$+ \cos n\omega t\sin m\omega t\, c\delta_n s\delta_m$$
$$+ \sin n\omega t\cos m\omega t\, s\delta_n c\delta_m\Big]$$

when you $\int$ over 1 period, $\int_{\text{period}} \cos n\omega t\cos m\omega t\, dt = \frac{T}{2}$ if $n = m \neq 0$.
$= T$ if $n = m = 0$
$= 0$ otherwise

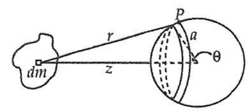
$\int_{\text{period}} \sin n\omega t\sin m\omega t\, dt = \frac{T}{2}$ if $n = m \neq 0$
$= 0$ otherwise.

Also,
$\int \sin n\omega t\cos m\omega t\, dt = 0$

So $\displaystyle \frac{1}{T}\int_{\text{period}} x^2\,dt = \sum_{n=1}^{\infty} \frac{D_n^2}{2}\left(\cos^2\delta_n + \sin^2\delta_n\right) + D_0^2$

$\displaystyle = D_0^2 + \sum_{n=1}^{\infty} \frac{D_n^2}{2}$ \ Parseval's thm

5-11.



The potential at P due to a small mass element dm inside the body is

$$d\Phi = -G\frac{dm}{r} = -G\frac{dm}{\sqrt{z^2+a^2-2za\cos\theta}} \tag{1}$$

Integrating (1) over the entire volume and dividing the result by the surface area of the sphere, we can find the average field on the surface of the sphere due to dm:

$$d\Phi_{ave} = \frac{1}{4\pi a^2}\left[-G\,dm\int_0^\pi \frac{2\pi a^2\sin\theta\,d\theta}{\sqrt{z^2+a^2-2za\cos\theta}}\right] \tag{2}$$

Making the variable change $\cos\theta = x$, we have

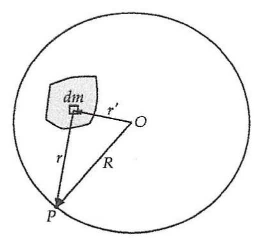
$$d\Phi_{ave} = -\frac{G}{2}dm\int_{-1}^{+1}\frac{dx}{\sqrt{(z^2+a^2)-2zax}} \tag{3}$$

Using Eq. (E.5), Appendix E, we find

$$\begin{aligned} d\Phi_{ave} &= -\frac{G}{2}dm\left[-\frac{1}{za}\sqrt{(z^2+a^2)-2za}+\frac{1}{za}\sqrt{(z^2+a^2)+2za}\right] \\ &= -\frac{G}{2}dm\left[\frac{-(z-a)+(z+a)}{za}\right] \\ &= -\frac{G}{z}dm \end{aligned} \tag{4}$$

This is the same potential as at the center of the sphere. Since the average value of the potential is equal to the value at the center of the sphere at any arbitrary element dm, we have the same relation even if we integrate over the entire body.

5-12.



Let P be a point on the spherical surface. The potential $d\Phi$ due to a small amount of mass dm inside the surface at P is

$$d\Phi = -\frac{Gdm}{r} \tag{1}$$

The average value over the entire surface due to dm is the integral of (1) over $d\Omega$ divided by 4π. Writing this out with the help of the figure, we have

$$d\Phi_{ave} = -\frac{Gdm}{4\pi}\int_0^\pi \frac{2\pi\sin\theta\,d\theta}{\sqrt{r'^2+R^2-2r'R\cos\theta}} \tag{2}$$

Making the obvious change of variable and performing the integration, we obtain

$$d\Phi_{ave} = -\frac{Gdm}{4\pi}\int_{-1}^{1}\frac{du}{\sqrt{r'^2+R^2-2r'Ru}} = -\frac{Gdm}{R} \tag{3}$$

We can now integrate over all of the mass and get $\Phi_{ave} = -Gm/R$. This is a mathematical statement equivalent to the problem's assertion.

5-21. (We assume the convention that $D > 0$ means m is not sitting on the rod.)

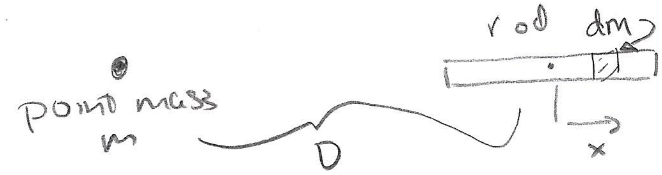
The differential force dF acting on point mass m from the element of thickness dx of the rod, which is situated at a distance x from m, is

$$dF = \frac{G(M/L)mdx}{x^2} \Rightarrow F = \int dF = \frac{GMm}{L}\int_D^{L+D}\frac{dx}{x^2} = \frac{GMm}{D(L+D)}$$

And that is the total gravitational force acting on m by the rod.

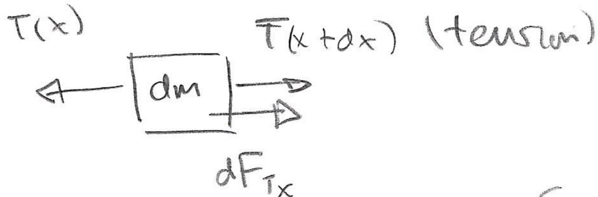
5-21b. Ignoring centrifugal force (as we did with tides)

$$dF_{Tx} = \frac{Gm\,dm}{D^3} \cdot x$$



$dm = \lambda\, dx$, λ = linear density of rod

We need to balance forces on mass dm.



so

$$T(x+dx) - T(x) = -dF_{Tx} = \frac{-Gm\lambda x\, dx}{D^3}$$

and

$$\int dT = \int \frac{-Gm\lambda x}{D^3}\, dx$$

$$T = -\frac{Gm\lambda}{2D^3} x^2 + \text{const.}$$

At ends, $T = 0$

so

$$0 = \frac{-Gm\lambda}{2D^3}\left(\frac{L}{2}\right)^2 + \text{const} \qquad \text{Const} = \frac{Gm\lambda L^2}{8D^3}.$$

Also $\lambda = \frac{M}{L}$. so

$$T = \frac{Gm}{2D^3}\left[\frac{ML}{4} - \frac{Mx^2}{L}\right]$$

max at $x=0$, zero at ends

5. $f = tq\dot{q}^2$

a) $\frac{\partial f}{\partial q} = t\dot{q}^2$ b) $\frac{\partial f}{\partial \dot{q}} = 2tq\dot{q}$ c) $\frac{\partial f}{\partial t} = q\dot{q}^2$

d) $\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial \dot{q}}\ddot{q} = q\dot{q}^2 + t\dot{q}^3 + 2tq\dot{q}\ddot{q}$

6. $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ $\frac{\partial f}{\partial y} = 0$ $\frac{\partial f}{\partial y'} = \frac{1}{2}x(1-y')^{-1/2} \cdot -1$

so

$$\frac{d}{dx}\left[\frac{x}{(1-y')^{1/2}}\right] = 0$$

$\frac{x}{(1-y')^{1/2}} = \text{Constant}$ $1-y' = cx^2$ $y' = 1-cx^2 = \frac{dy}{dx}$

$$y = \int (1-cx^2)\,dx = x - cx^3 + D$$

when $x=1, y=1$ $1 = 1-c+D \rightarrow c=D$

when $x=3, y=2$ $2 = 3-27c+D$

$1 = 2 - 26c$ $-26c = -1$ $c = +1/26. = D.$

7. $\frac{\partial f}{\partial y} = 0$ $\frac{\partial f}{\partial y'} = 2xy'$ $\therefore \frac{d}{dx}\left[2xy'\right] = 0$ $2xy' = \text{Constant}$

$\frac{dy}{dx} = \frac{c}{2x}$ $\int dy = \int \frac{c \cdot dx}{x}$ $y = c\ln x + D.$

8. $ds^2 = r^2 d\theta^2 + dz^2$. Write $z = z(\theta)$. We take $r = 1$.

We want to minimize $\displaystyle\int ds = \int d\theta\, \underbrace{(1+z'^2)^{1/2}}_{f(z,z';\theta)}$ $\qquad z' = \frac{dz}{d\theta}$

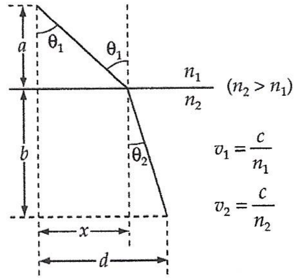
so $\frac{\partial f}{\partial z} = 0 \qquad \frac{\partial f}{\partial z'} = \frac{1}{\cancel{2}}(1+z'^2)^{-1/2}\cdot \cancel{2}z'$

so $\frac{d}{d\theta}\left(\frac{\partial f}{\partial z'}\right) = 0 \qquad \frac{\partial f}{\partial z'} = \text{constant} = c_1 = \frac{z'}{(1+z'^2)^{1/2}}$

$1 + z'^2 = c_2 z'^2 \rightarrow z'^2 = \text{constant} \qquad z' = \text{constant} = \frac{dz}{d\theta}$

Helix.

6-7.



The time to travel the path shown is (cf. Example 6.2)

$$t = \int \frac{ds}{v} = \int \frac{\sqrt{1+y'^2}}{v} dx \tag{1}$$

Although we have $v = v(y)$, we only have $dv/dy \neq 0$ when $y = 0$. The Euler equation tells us

$$\frac{d}{dx}\left[\frac{y'}{v\sqrt{1+y'^2}}\right] = 0 \tag{2}$$

Now use $v = c/n$ and $y' = -\tan\theta$ to obtain

$$n \sin\theta = \text{const.} \tag{3}$$