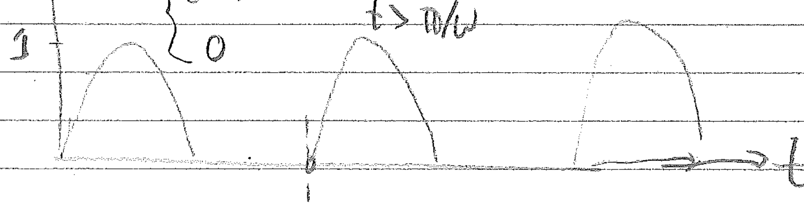
Linear

More about fourier series + Harmonic oscillator.

"A linear oscillator responds only at the driving frequency, not at its natural frequency."

(It responds more strongly to a driving force near its natural frequency than far from it.)

Consider this force $F = \begin{cases} \sin \omega t & t < \pi/\omega \\ 0 & t > \pi/\omega \end{cases}$



The fourier series for this is

$$F(t) = \frac{1}{\pi} + \frac{1}{2}\sin \omega t + \sum_{n=1}^{\infty} \frac{2}{\pi(1-4n^2)} \cos 2n\omega t$$

What is the response of a L.H.O. to this force?
It is the sum of the responses to each term.

Each term looks like $A\cos(\omega_n t + \phi_n)$

1st term $\omega_n = 0$, $\phi_n = 0$, $A = 1/\pi$
2nd term $\omega_n = \omega$, $\phi_n = -\pi/2$. $A = 1/2$
$\sum_n$ terms: $\omega_n = 2n\omega$, $\phi_n = 0$, $A = \frac{2}{\pi(1-4n^2)}$

Solution is $x = D\cos(\omega_n t + \phi_n - \delta_n)$ $\delta_n = \arctan\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$

$$D = \frac{A}{((\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2)^{1/2}}$$

Some examples on slides.

Note response to $\omega = \omega_0$, $\beta = 0.1\,\omega_0$.

Response to full sine wave would have $D = \dfrac{1}{\left((\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right)^{1/2}}$

$= \dfrac{1}{2\beta\omega} = 5.$

i.e. 10 units (meters) peak to trough.

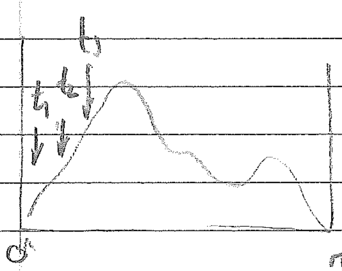
The actual response is half that. The response is dominated by the $\sin \omega t$ term in the Fourier Series, which has a pre-factor of $1/2$.

Note also that x peaks a quarter cycle after F. $\delta = \frac{\pi}{2}$

ANY NON-SINUSOIDAL FUNCTION ($t = -\infty \to +\infty$) CONTAINS MULTIPLE FREQS.

Sines and cosines are called basis functions for F. Series

Why do we say they are "orthogonal"



A function defined on this interval has a value at each point t

We can think of this function as an ∞-dimensional vector.

The coordinate "axes" are t_1, t_2, t_3, t_4, t_5 etc. on.

So "$\cos\omega t$" is a vector in ∞ dimensional space.
So is "$\sin\omega t$".

Two vectors are $\perp$ if dot product $= 0$.

$\overrightarrow{\cos\omega t} \cdot \overrightarrow{\sin\omega t} = \sum_i \cos\omega t_i \cdot \sin\omega t_i + \cos\omega t_2 \sin\omega t_2 + \ldots$

that's ∞, of course, but if we scale each product by dt, we get a finite answer.

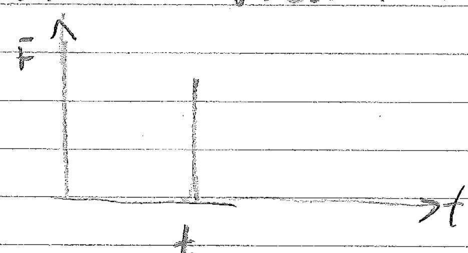
$$\cos \cdot \sin = \int_{\text{period}} \cos\omega t \, \sin\omega t \, dt = 0$$

So: cosines and sines are vectors. $\cos(n\omega t)$, $\sin(n\omega t)$ form an infinite set of basis vectors.

To solve the H.O. you just need to know how it responds to your basis vectors. Then express the F_d in terms of basis vectors.

Could we use different basis vectors? Yes!

Why not use this?



an impulse @ t_1?

We call this $\delta(t_1)$ "delta function"

these are trivially orthogonal $\int \delta(t_1)\delta(t_2)\,dt = 0$ for $t_1 \neq t_2$.

What is H.O. response to an impulse? Book uses

Easier. After an impulse $v = \dfrac{J}{m} = \dot{x}(0)$

Soln to undriven oscillator with this initial cond

$$x = Ae^{-\beta t}\cos(\omega_1 t - \delta)$$

$x(0) = 0$ $\quad \delta = \pi/2$. so $\quad x = Ae^{-\beta t}\sin\omega_1 t$

$\dot{x}(0) = \dfrac{J}{M} = \omega_1 A$ so $A = \dfrac{J}{M\omega_1} = \dfrac{1}{M\omega_1}$ for a unit impulse.

For an impulse at time t' we just shift the response

$$G(t,t') = \frac{1}{m\omega_1} e^{-\beta(t-t')} \sin \omega_1 (t-t') \qquad t \geq t'$$

$$= 0 \qquad t < t'$$

Now, any force $F(t)$ is really just a series of impulses, and because of linearity, we can add the response from $F(t_1)$ and $F(t_2)$ and $F(t_3)$ etc

i.e.

$$\boxed{x(t) = \int_{-\infty}^{t} F(t') G(t,t') \, dt'}$$ Green's function.

So Green & Fourier are <u>same idea</u> with different bases!