Making a Stretched String

Is it not strange that sheep's guts should hale souls out of men's bodies?

—William Shakespeare, *Much Ado about Nothing*

A stretched string is in some ways like many other vibrating objects, but in other ways it is very special, making it an ideal source of vibrational energy for musical instruments. The string becomes a drive for the body or sounding board, which produces almost all the sound.

Our object in this chapter is to systematically construct a string under tension. We do so by treating the string as little beads connected to each other by a massless filament under tension. If our only purpose were to understand string vibrations, we would not bother to build up a string meticulously from little beads under tension. But this way, we learn how assemblies of independent parts, connected by forces, can vibrate. We discover how all the parts can collectively conspire to produce simple and nearly universal oscillations and waves of choreographed motion. A number of new principles will emerge, all extremely useful for a deep understanding of sound.

In the next pages, we encounter standing waves, traveling waves, frequencies of vibration and the dependence of frequency on tension, mass, length, superposition, periodic and aperiodic vibrations, and the tones that emerge from different types of vibration. The suggested interactive applets and sound analysis tools provided are key to reaching a deeper understanding.

A string is essentially a one-dimensional object. We will find that the ideal string possesses a perfectly *harmonic* (equally spaced in frequency; all frequencies an integer multiple of the fundamental) series of modes. Real strings have some thickness, ruining the perfectly harmonic frequencies we shall derive here. Far from a bug, the mistuned partials of real strings are

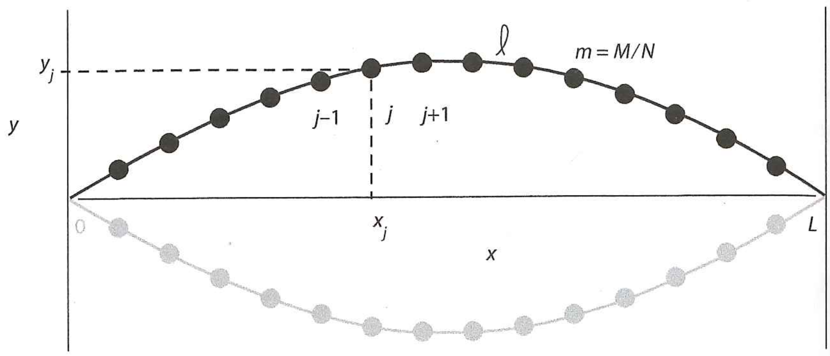


Figure 8.1
The lowest vibrational mode of a string of *N* beads, connected by a filament under tension. The gray outline shows the lower extent of the oscillation; the black is the upper extent. At both extremes, the beads are all momentarily at rest.

a feature that we have grown to love and expect. A piano sounds artificial without them.[1]

We start with a single bead of mass M under tension, held by two massless elastic filaments. This bead is next split into two beads, each of half the weight, dividing the filament into three equal lengths, and eventually into N equally spaced beads of mass $m = M/N$, distributing the beads uniformly over the same length of string, making a "necklace." As N gets large, the assembly starts to look like a continuous object for most purposes. As we build up a string from ever more and ever lighter beads, the string is always kept the same length, L. The beads get closer together, a distance $\ell = L/(N+1)$ between beads. The situation is depicted in figure 8.1. We start with a single bead of mass M tied between two walls a distance L apart by stretched massless filaments. We consider only up-and-down oscillation of the bead in the plane of the paper. After we see how this moves (a harmonic oscillator), we will divide the bead into two beads each of mass $M/2$, keeping the total mass the same. Now there will be two independent ways, or *modes*, for the two beads to vibrate, each with a different frequency. Each of these modes is itself a harmonic oscillator. The oscillation involves a choreography of more than one bead, but nonetheless there is a mass being displaced and a force of resistance proportional to the displacement—the key ingredients for a harmonic oscillation.

Each time a bead is added, we rebuild the whole necklace and find N modes of vibration, each with a unique shape and frequency. There is a pattern that develops: for N beads, the first $N-1$ modes pay homage, so to speak, to all the modes of $N-1$ beads, mimicking them as closely as possible, but the highest frequency mode in the list is always a "new" mode.

[1]The partials of a piano string are slightly sharp compared to the integer multiples of the lowest partial. This effect is weak for the first few partials, but grows more important for the higher partials. The higher partials possess shorter wavelength oscillations on the string, bending it in tighter curves. If the diameter of the string gets to be noticeable on the scale of the wavelength on the string, it starts to act a little like a bar rather than a string, stiffening its bending resistance and raising its frequency. We will ignore the finite thickness in this chapter.

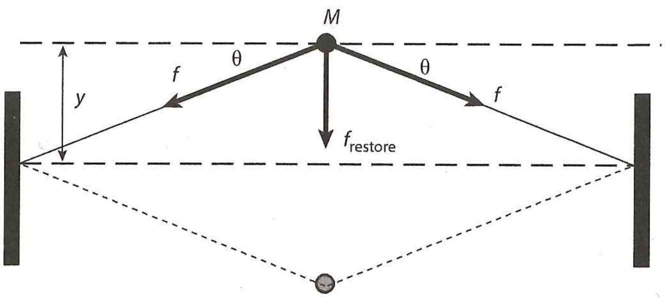


Figure 8.2
Forces on a single bead of mass M, held by two elastic filaments under tension and attached to rigid walls at either end. The downward force, $f_{restore}$, is proportional to y for y not too large.

As we build up more complex objects from simple ones, we will see *how they become simple again* when they vibrate in *collective* modes, where their constituent parts act in concert. We rely in part on firsthand experience with Paul Falstad's immediately accessible *Loaded String* applet and a sound recording and analysis tool such as *Audacity*, both free on the Internet and linked on whyyouhearwhatyouhear.com.

8.1 Single Bead

We start with a single bead held by filaments between two walls with tension. The bead is considered a mass point with no internal structure of its own. The tension is supplied by elastic filaments that have no mass of their own.

Tension and Force

Figure 8.2 shows a single bead held by two elastic filaments under tension. Pulling vertically on the bead (we call the displacement y, which can be positive or negative), a restoring force $f_{restore} \propto y$ is felt pulling the bead back toward its equilibrium, horizontal position ($y = 0$). The bead is under tension being pulled to the left by the left portion of the filament and to the right by the right portion. Tension is a force; it has a magnitude and a direction. A force communicated by a string or filament is necessarily aligned along it. A filament is strong only along its own length, it cannot support any force perpendicular to itself. For the single bead at rest, the two forces, one from each filament, are equal in magnitude and opposite in direction, so they cancel. Then, according to Newton's second law of motion, there is no movement, no acceleration.

If the bead is displaced vertically, part of the force from each side of the filament points down, as seen in figure 8.2. For small displacements, the magnitudes of the forces from each filament to either side of the bead are the same as before, but now the forces are no longer exactly opposite. We represent the force by an arrow; we must always point the arrow along the filament. We take the length of the arrow proportional to the magnitude of the force.

The vertical part of the force does not find a canceling force; this is the restoring force. When the bead is released, the bead accelerates downward according to Newton's law, $f_{restore} = Ma$, where $f_{restore}$ is the net force on the bead, M is the mass, and a is the acceleration, in the direction of the net force. If the bead was first at rest in the position shown, it will start moving down at increasing speed. (Gravity acceleration is normally very weak compared to acceleration due to tension; we ignore gravity here.) As the bead moves down, the angle made by the two filaments decreases, vanishing as the bead passes the midpoint. But the bead is moving fast: it overshoots into negative territory, now feeling a restoring force in the opposite direction, slowing it down. It comes to a stop at the position shown by the dotted line at the bottom of figure 8.2.

It is important to remember that the bead is always under tension from the filaments. The tension in each filament increases only very slightly (we ignore the effect) as the bead is displaced. It is the change in the *direction* of the force that drives string vibration; it remains always aligned with the filament.

Figure 8.2 can be used to show that the force is proportional to the displacement, y:

$$f_{restore} = Ma = -ky, \tag{8.1}$$

where k is the constant of proportionality, and the minus sign is there because the force is toward negative y when the displacement is positive. It is very easy to show, with a little trigonometry, that $k = 4T/L$, where T is the tension, and L is the length of the string. The bead and filament system is thus a member of a very large and important class of vibrating objects mentioned earlier in connection with equations 3.1 and 3.3: the harmonic oscillator. The distinguishing feature of such vibrations is that the force on the object is proportional to the displacement, and in the opposite direction. A tensioned filament makes a good linear spring against displacement of the bead.

The Motion of the Bead

The motion of the bead is

$$y(t) = y_0 \cos(2\pi f t + \delta); \;\; f = \frac{1}{2\pi}\sqrt{\frac{k}{M}} = \frac{1}{2\pi}\sqrt{\frac{4T}{LM}}, \tag{8.2}$$

where y_0 is the amplitude of oscillation, (the maximum displacement away from 0), and δ is a phase shift, both of which depend on the initial conditions. Initial conditions are the initial position and velocity that the bead is given. The simplest case, sufficient for most of our purposes, is to assume that the bead was displaced by an amount y_0 and at rest at time $t = 0$. Then y_0 is the initial displacement and $\delta = 0$. The sinusoidal function has appeared again.

8.2 Two Beads

The case of two beads is a watershed: a reader comfortable with the motion of two beads is "good to go" to any number of beads and much more complex objects. We again consider only vertical motion for each bead. What can happen for two beads? Is sinusoidal motion obsolete for more complex objects?

With a little more algebra than we care to go into here, we can show there are now *two* modes, each with a different frequency and a unique "shape," which we shall define. Each mode is *sinusoidal* in the following way: pick any bead and follow its vertical motion over time assuming just one mode is excited and therefore just one frequency is present. That motion will be exactly sinusoidal (or it will not be moving at all—that is, a nodal bead) at the frequency of the mode.

Box 8.1

Working with *Loaded String*

Paul Falstad's *Loaded String* applet follows exactly the program we have just set out. The number of beads (called *loads*) can be set from one to hundreds with a slider. It is strongly suggested that you experiment with this applet. The reward for the time spent will be intuition and understanding for complex vibrating objects.

Starting with one bead, notice the frequency increase as the tension is increased. Leave external forces and damping off; we get to those in the next chapter. The sound of a harmonic oscillator is the pure but boring sinusoid. There is one mode, one frequency.

When the number of beads is two, the mass of each bead is $M/2$. The filaments are shorter, being divided into three equal segments instead of two, but the tension is not changed. In the *Loaded String* applet, there are two "stalks" at the bottom left of the screen for two beads; each controls the amount of its corresponding pure, sinusoidal mode. Click the Clear button, and then drag the leftmost stalk up. The two beads rise together, as in the second row of figure 8.3. This is the shape of the lower frequency of the two modes. The way we define shape, pulling the stalk higher or even (with the phase stalks) making the beads go below the line, does not change the shape; rather, this is changing the amplitude of the mode. Each mode has a unique shape.

Clear the Stop option, and watch the subsequent oscillation. Note that the initial position is regained periodically. The shape remains the same

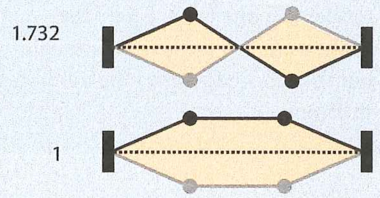


Figure 8.3

The two possible modes of a two-bead system, with the lowest frequency mode assigned a frequency of 1. The beads in black are shown at rest in their deformed positions. The gray beads and lines show the limit of the excursion at one half period; this is a symmetric reflection about the horizontal dotted line.

Working with *Loaded String* *(continued)*

throughout the oscillation: at each time, the beads are in a position that could have been obtained by multiplying the initial shape by a constant. The motion of each bead is sinusoidal, all moving at the same frequency starting from different positions: the result is a concerted, choreographed dance of the two masses. Click Play, and you will hear a pure sine tone at the frequency of oscillation and see the periodic motion of the two beads.

Click Clear again, and raise the second stalk. Now a different type of vibration is seen, having a new shape and a higher frequency. Notice that the second mode is oscillating more rapidly than the first. If you look at how the filaments are oriented relative to the masses, you can see why the force is higher and therefore the acceleration is greater than in the first mode, leading to higher frequency and a larger force constant. The relative frequencies of these two modes is accurate, although the frequency of the lower mode has been set equal to the frequency of the mode with one bead. Again the motion of each atom is sinusoidal, but the "choreography" the atoms is different.

If this much is plausible to you, there is only one more aspect to discuss: combinations of the two modes simultaneously.

Compound oscillation, wherein more than one mode is excited at the same time, can easily be produced with the Mouse = Pluck String option. Or you can pull up two (or more, for more beads) stalks at the bottom-left corner, creating a combination of pure modes. Last, you can select the Mouse = Shape String option and make any initial shape for the string.

Exciting more than one mode by plucking will create a shape distinct from any of the "pure mode" shapes discussed earlier. Each excited mode oscillates sinusoidally, but since they have different frequencies the modes are combining with a relative phase of their respective oscillations that is continuously changing over time. As a result, the shape taken by the two beads changes with time. The two pure mode frequencies are not simply related to each other, and as a result the combined choreography of the beads after such a "pluck" is not periodic. Selecting Play may fuse the two frequencies to give a single, somewhat unmusical tone, or the two component sinusoids may stand out instead; this will depend somewhat on the listener and the context (see chapter 23).

The two modes, in various combinations differing in the amplitude of each mode and their relative phase, can describe *any* initial shape for two beads that is possible. Figure 8.3 makes clear how two pure modes combine. As you pluck the beads, thus shaping the "string," the applet automatically finds the right combination of the pure modes to produce your pluck and reveals that combination as a set of amplitudes in the lower left of the panel. If you hover the mouse pointer over one of the stalks, the stalk turns yellow, and the appropriate pure mode panel above also turns yellow, revealing the pattern of vibration associated to that stalk—that is, the pure or "normal" mode, as it is called. If you hover the mouse pointer over the stalk while the application is animating the vibration, the phase and amplitude of the individual normal mode taking part in the combination is shown. It is difficult to imagine a more instructive applet to illustrate these points.

The Sinusoid Reigns Supreme

For two beads, we have two modes of vibration, instead of one. This leads us to suspect there will be N different modes for N beads. A *pure mode* is identified as an oscillation in which the parts of the object (beads in this case) moved sinusoidally with a single frequency. A stylus attached to any one of the beads as it oscillates in one of the pure normal modes traces out a sinusoid on graph paper moving from right to left (see figure 8.4 for the

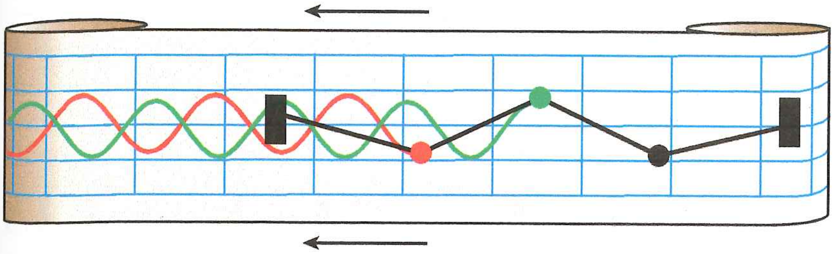


Figure 8.4
In a pure mode, individual beads move sinusoidally and share the same frequency. The chart of any given bead's motion is a sinusoid, just as it is with a single bead. This is true no matter the number of beads.

case of three beads). During the oscillation of a pure mode, the shape does not change, in the sense defined earlier.

Now that we have come to more complex objects involving several parts, we see that the *the sinusoid still reigns supreme, exactly describing the motion of each part of the object.* To be specific, the ith bead in the nth mode has a displacement in the y direction that increases

$$y_{n,i}(t) = a_{n,i} \sin(2\pi f_n t). \tag{8.3}$$

Note that all the beads for a given mode n share the same frequency f_n, but each in general has a different amplitude $a_{n,i}$. The same sinusoidal function multiplies all the amplitudes for a given mode n, showing that indeed the shape is retained during the oscillation.

Returning to the two-bead case, we see that the lower frequency mode, wherein both of the beads move together in the same direction, is the analogue of the only mode that exists for one bead. This is the first example of the fact that N beads recapitulate all the modes of the $N - 1$ bead case, and add one new mode, the one at the highest frequency.

8.3 Three Beads

Moving to three beads in *Loaded String*, we see that indeed the two modes of the two-bead case are recapitulated by the first two modes of the three-bead case. The third mode is new. (You can always excite a pure mode by clearing the screen and then pulling up one of the stalks.) All of these modes and their analogs for more beads are themselves harmonic oscillators: they oscillate with different frequencies for different modes, but each mode retains the same shape throughout the oscillation. The sinusoidal motion of individual beads is made explicit in figure 8.4, where a red bead and a green bead are tracing out their motion on moving graph paper.

A given pure mode, with only one stalk raised, will have a pattern of vibration involving most of the beads, except for the occasional bead that isn't moving. Such a bead finds itself at a node of the vibration, a place where the periodic, sinusoidal undulation is actually quiescent.

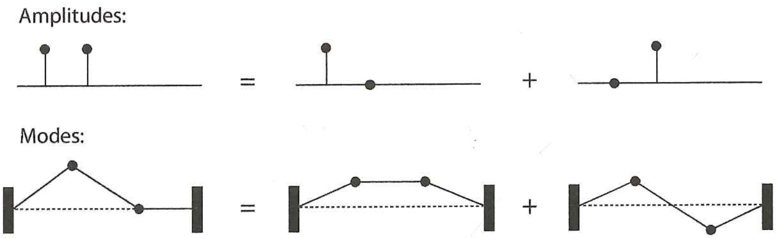


Figure 8.5
The addition of two normal modes depicted as abstract amplitude (top) and as the literal addition of the mode displacements (bottom). In the latter case, the linear addition of the vertical displacements of the two normal modes on the right gives the shape on the left, which is made up of both modes. For example, the rightmost bead is up in the first mode, but down by the same amount in the second, so the addition of the two modes puts the bead at zero displacement, as it is shown on the left. If you turn on the sound, you hear two frequencies; these, however, are not musically related—for example, $f_2 \neq 2f_1$, or any such simple ratio. Note that the oscillation of the shape (slow it down if it is too fast) reveals that it is *not* periodic; it does not recover its shape in a regular way.

8.4 Combining Modes

Any shape can be reproduced as a combination of all the pure modes, if the correct amplitudes are used. We illustrate the superposition in the two-bead case in figure 8.5; it is easy to see exactly how the new shape is created out of two normal modes. A single bead will oscillate *at* N *different frequencies at once* if *N* modes are excited.

The idea of combining purely sinusoidal modes to make complex nonsinusoidal and even nonperiodic motion is a crucial one, a key to the vibration of real objects under real circumstances. The resulting combined motion is not usually periodic, even though the component modes are, and is a sum of sinusoids whose frequencies are normally not *commensurate* (do not bear a simple integer or rational ratio to each other).

8.5 More Beads

In the case of the string that we are now studying, sinusoids make another appearance of a different sort. For sufficiently many beads, the *shape* of the individual modes starts to trace out a recognizable sinusoid. Even in the cases with relatively few beads, the beads fall on a curve belonging to each mode; this curve is exactly the sinusoidal one followed for an infinite number of beads.

For three beads (figure 8.6), the highest frequency mode is a new shape compared to the two found for two beads. This shape is recapitulated ever more smoothly for 4, 5, . . . beads; its sinusoidal shape is clearly apparent with 4 beads (see figure 8.7). Try these cases in the *Loaded String* applet. Choosing 14 to 20 beads or so, clear all of the oscillations, and then set the amplitudes by lifting the stalks of only the first three or four modes. Click Play, and you will hear a near-musical tone. The tone is not quite periodic because the component partials are a little flat compared to equal spacing. Note that the shape you have created out of the combination of four modes is nearly repeating itself periodically, with the period being that of the lowest frequency mode. Fifteen or 20 beads is not quite enough to get the first few modes to vibrate at frequencies that are almost exactly integer multiples of the lowest frequency mode. Using 100 beads or more, the first four or five modes are nearly equally spaced multiples of the fundamental, lowest mode frequency.

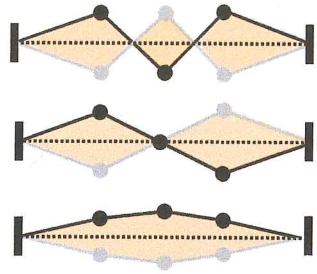


Figure 8.6
The three modes of a three-bead system, with frequency ratios relative to the lowest frequency mode.

The Sound and Spectrum of a Pluck

If the mouse is used to pluck the string of N beads in one place, all the modes are excited, including the higher modes. (The higher frequencies present can be easily heard.) By the way, this is a very important lesson in sound production, musical or not: if objects are struck locally, many higher frequency modes as well as lower frequency modes will be excited. If, however, the object is struck with a soft, rounded hammer, the tendency will be to excite only lower frequency modes. It is easy to see why: in order to describe the deformation owing to a blow with a sharp object, the short wavelengths will be required because they oscillate faster and are essential to reproduce sharp changes in the shape of the object. This can easily be checked in *Loaded String* using the Shape String option and plucking a single bead versus an initial deformation that involves displacing several adjacent beads in such a way as to trace out a smooth shape. The proof of the difference will be heard as well as seen in the height of the amplitude stalks.

The tone produced by a sharp pluck is a little sour and is especially defective at the higher frequencies due to the presence of high modes that are flat compared to their counterparts in an ideal string. The tension can be adjusted to raise or lower the pitch, but the note is still sour. (The fact that there is a definite pitch at all, and exactly what that pitch is, since the time signal is not periodic due to inharmonic partials, is an intriguing and subtle question, to which we will devote chapter 23.)

It is instructive to actually measure the spectrum of frequencies coming out of a laptop speaker when the sound is turned on in *Loaded String*. Most computers can record themselves with their own built-in microphone at the same time they are producing sound, or more directly with software

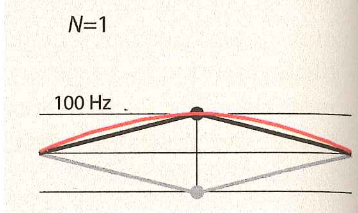


Figure 8.7

Building up a continuous string starting from beads connected by filament. All the possible modes for one through four beads are shown explicitly. The red curve is the analogous sinusoidal mode for an infinite number of beads. Note that the beads always fall on the $N = \infty$ smooth sinusoidal shapes. As $N \to \infty$, the mode separations all become 100 Hz. The actual mode separations for $N = 2, 3,$ and 4 beads are shown in red. In all cases, the total mass of all the beads M is held fixed. Each mode is a harmonic oscillator, and once excited it executes sinusoidal motion in isolation of its companion modes with the same number of beads. For N beads, there are N different modes of different frequency. For a small number of beads, the various modes are quite inharmonic (unequally spaced), and the beads, if plucked, sound more like a bell or chime than a plucked string. As the number of beads grows, the lower modes become more nearly evenly spaced, and hitting the string with a wide, smooth hammer (which excites only lower modes) results in a periodic string-like tone.

utilities. This recorded sound can in turn be analyzed. The sound files, waveforms, and spectra for two different kinds of pluck of a 30-bead string are given at whyyouhearwhatyouhear.com. A 7-bead string was recorded and analyzed in figure 8.8.

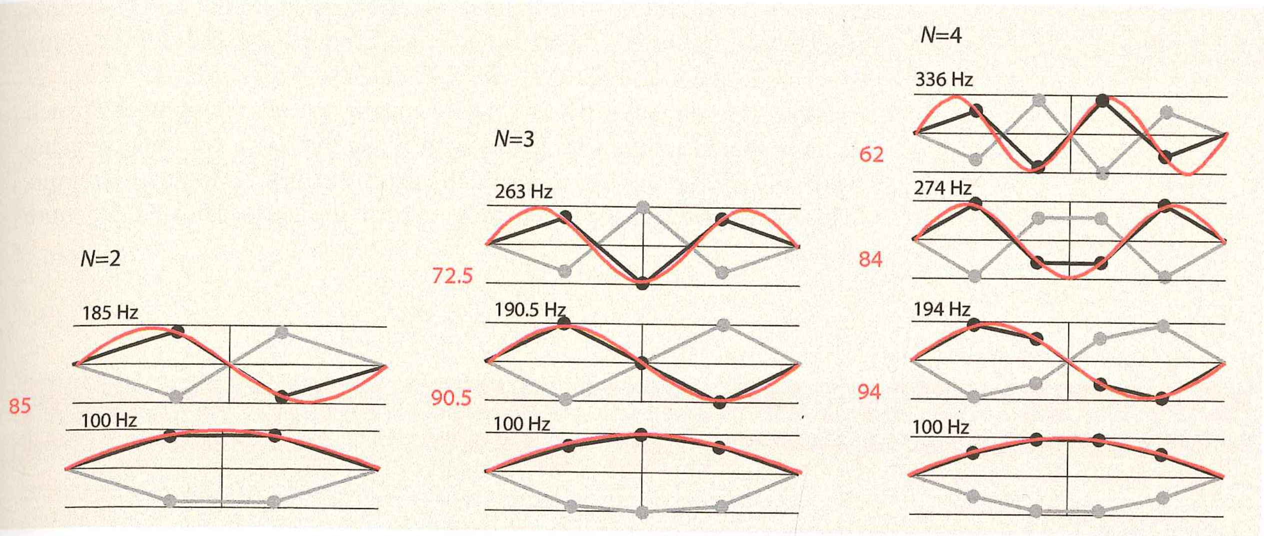
The *inharmonic partials* (inharmonic meaning that the partial frequencies are not equally spaced) that we find for relatively few beads is typical of vibrating systems. If you bang on a piece of metal, you are not likely to hear a pleasing periodic tone.[2] The string with N beads is inharmonic in a systematic way, with the spacing between adjacent frequencies growing smaller at the top of the spectrum. Typical objects (a string is certainly not typical) will tend to have more jumbled, seemingly unsystematic spectra. We will meet systems with inharmonic partials many times again; they are the norm, not the exception, for vibrating objects. Bells, chimes, and so on have inharmonic partials, but they are carefully crafted through thickness and shape adjustments to give a pleasing tone and a desired pitch (which, as we will see, need correspond to none of the actual mode frequencies).

In spite of the clang one gets by bashing a piece of metal dangled on the end of a string, it is good to remind ourselves that this noise is no more and no less than a superposition of sinusoids, each an atom of sound, as pure as can be.

For N beads, an analysis more detailed than we care to enter into here shows that the exact frequencies are given by

$$f_n = \frac{N}{\pi}\sqrt{\frac{T}{M\cdot L}}\,\sin\left(\frac{n\pi}{2(N+1)}\right) = \frac{N}{\pi}\sqrt{\frac{T}{\rho\cdot L^2}}\,\sin\left(\frac{n\pi}{2(N+1)}\right), \tag{8.4}$$

[2]There are exceptions; see the discussion of Belleplates, section 15.7.



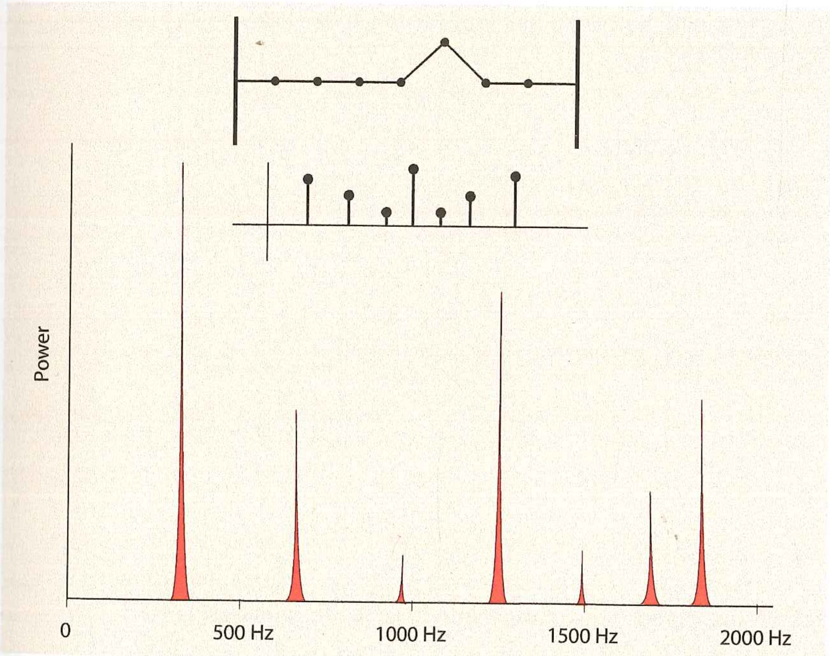


Figure 8.8

Seven-bead string as studied in Falstad's *Loaded String* applet and analyzed in *Amadeus*. The initial pluck is shown at the top, and the resulting power spectrum stalks according to the applet are shown just below. The sound was played through small speakers and recorded by microphone at the desktop and then analyzed in *Amadeus*. The spectrum corresponds nicely, and clearly shows the smaller spacing of the mode frequencies at the top of the spectrum.

where M is the total mass of the beads (each bead being of mass M/N), T is the tension, L is the length of the assembly of beads, and $n = 1, 2, \ldots, N$ is an index that tells us which mode we are speaking of, the lowest frequency being $n = 1$. We have introduced the density of the string, $\rho = M/L$, which has units of mass per length.

We are not concerned with the fine details of this formula for the frequencies, but there are two qualitative aspects that are quite important. First, we notice that because of the sine function in this formula the mode frequencies cannot be equally spaced—that is, they cannot be a harmonic series: the *argument* of the sine function is equally spaced, but applying the sine to the argument destroys the equal spacing. Second, for large N the lowest frequencies are *very nearly* equally spaced; the number of such equally spaced frequencies grows with N, so that for a continuous string with an infinite number of beads the spectrum will be totally harmonic. This follows from the fact that $\sin x \approx x$ for small x (see box 8.2 for more details).

Box 8.2

Spectrum for a Large Number of Beads

To show that the spectrum becomes equally spaced in the limit of a large number of beads, we notice that the argument of the sine function, $n\pi/2(N+1)$, becomes small as the number of beads N gets large, for any fixed n. A very useful approximation to the sine function for small y was given already in chapter 3:

$$\sin y \sim y. \tag{8.5}$$

The approximation is better and better, the smaller y is. (Try it on a calculator, but make sure you are using radians and not degrees.)

Making this approximation in equation 8.4 leads to

$$f_n = \frac{N}{\pi}\sqrt{\frac{T}{M \cdot L}}\left[\frac{n\pi}{2(N+1)}\right] = n \cdot f_1, \tag{8.6}$$

where, since $N/(N+1)$ is very close to one for large N,

$$f_1 \sim \frac{N}{2(N+1)L}\sqrt{\frac{T}{\rho}} \rightarrow \frac{1}{2L}\sqrt{\frac{T}{\rho}}, \tag{8.7}$$

thus

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\rho}}, \tag{8.8}$$

where by the symbol $\rightarrow$ we mean that N is large, which makes $N/(N+1) \rightarrow 1$.

Equation 8.8 captures the Mersenne laws of vibrating strings: their frequency is inversely proportional to the length of the string, proportional to the square root of the tension, and inversely proportional to the square root of the density of the string per unit length.

Again without proof, we give the formula for the y displacement of each bead in each of the N modes for N beads:

$$y_{jn0} = \sqrt{\frac{2}{N}}\sin\left(\frac{jn\pi}{N+1}\right). \tag{8.9}$$

The constant $(2/N)^{1/2}$ is fixed for a fixed number of beads, it just sets the overall amplitude. Equation 8.9 gives the amplitude for the jth bead in the nth mode. This is the solution we see when we run the *Loaded String* applet.

Noticing that the position of the jth bead is given by $x_j = jL/(N+1)$, we can write

$$y_{jn0} = \sqrt{\frac{2}{N}} \sin\left(\frac{n\pi x_j}{L}\right). \tag{8.10}$$

This form highlights the fact that the y displacement is a sinusoidal function of the position of the jth bead. The sine function is seen to play a crucial role in the shape of the string, even for a finite number of beads.

The all-important relation $f\lambda = c$, where f is the frequency, λ is the wavelength, and c is the speed of the wave, can always be used to determine the third variable if two others are known. Now we know the frequency f_n of the modes, and the wavelength λ_n is determined from equation 8.10: $n\pi\lambda_n/L = 2\pi$, or $\lambda_n = 2L/n$, $n = 1, 2, \ldots$. Then, from equation 8.8 and $f\lambda = c$, we have

$$f_n\lambda_n = \left(\frac{nT}{2L\rho}\right)\left(\frac{2L}{n}\right) = \sqrt{\frac{T}{\rho}} = c. \tag{8.11}$$

We have found that the speed of waves on the string is given in terms of the tension and the mass density as $c = (T/\rho)^{1/2}$, independent of the mode number, and therefore is independent of the wavelength and frequency, just as with sound in air. Since $(T/\rho)^{1/2} = c$, we can write equation 8.8 as

$$f_n = \frac{nc}{2L}. \tag{8.12}$$

We can now write equation 8.8 as

$$f_n = \frac{nc}{2L}. \tag{8.13}$$

There is much information in this deceptively simple formula. The frequencies are equally spaced (harmonic). The speed is independent of the frequency. The Pythagorean rule, that frequency is inversely proportional to length at the same tension, is also contained in equation 8.13.

Music, no matter from what society, does not by any means restrict itself to instruments with equally spaced harmonics. To understand the full gamut of musical sound production, as well as sound made by myriads of other sources for other reasons, we need to understand the vibrations of generic objects as much as we do those of strings. We have already discussed sound production by objects such as surfaces, but we have not discussed the natural frequencies of vibration of objects like plates and bells. We do so in chapter 15.

8.6 Putting Shape and Time Together

The complete motion for the jth bead in the nth mode is

$$y_{jn}(t) = y(x_j, t) = a_n \sin\left(\frac{n\pi x_j}{L}\right)\cos(2\pi f_n t + \delta_n). \tag{8.14}$$

When $N \to \infty$, this becomes

$$y_n(x, t) = a_n \sin\left(\frac{n\pi x}{L}\right)\cos(n\pi c/Lt + \delta_n), \tag{8.15}$$

which can be written more simply, by defining $k_n = n\pi/L$ and $f_n = nc/2L$, as

$$y_n(x, t) = a_n \sin(k_n x)\cos(2\pi f_n t + \delta_n). \tag{8.16}$$

δ_n is a phase shift that in general we need to keep track of in order to get the correct waveform. This is in the form of a shape function, $\sin(k_n x)$, multiplied by a time function, $\cos(2\pi f_n t + \delta_n)$. The shape of the string in a pure normal mode is sinusoidal, in the sense mentioned earlier, and the time dependence is sinusoidal as well. In the nth mode, each bead oscillates at exactly the same frequency, frequency f_n. The period of the oscillation T_n, is simply the inverse of the frequency, $T_n = 1/f_n$. After this period, the mode necessarily has returned to its initial position.

Equation 8.13—$f_n = nc/2L$, with $c = (T/\rho)^{1/2}$—is full of implications:

- The string with fixed ends has many distinct modes, each of which is a sinusoidal vibration corresponding to a collective harmonic motion.
- The frequencies of the modes for a given string are equally spaced, being multiples of $f_n = nc/2L$.
- Since $f_1 \propto 1/L$, if the string length is halved, all other things held constant, the frequencies will double—that is, go up an octave. Similarly, a ratio of lengths of 2:3 yields a perfect fifth interval 3:2, and so on.
- The frequency is doubled by a fourfold increase in tension T.
- The frequency is halved by a fourfold increase in string density (mass per unit length). This can be done by using heavier material and/or by making the string thicker.
- Because frequencies are equally spaced, combinations of two or more modes will oscillate periodically, with the displacement of a portion of the string becoming a sum of sinusoids of different frequency.

We have seen how a complex object consisting of many parts (beads, in this case) can behave simply and coherently, by vibrating in unified ways we call normal modes. *Each normal mode is an independent harmonic oscillator—every bead or part of the string oscillating sinusoidally in time.* Even a million beads act as a whole, with thousands of beads making a choreographed sinusoidal motion in each mode. The simplest mode is the lowest frequency one consisting of half a sine wave moving up and down with all beads acting in concert. In this way, a kind of simplicity has been reborn from the potential complexity of having so many beads.

Humanity has enjoyed stringed instruments for perhaps 20,000 or 100,000 years. It is understandable that we take the perfections of the stretched string for granted, but it is a miraculous gift that they are harmonic. It is worth stating that inharmonicity of strings would not be fixable with frets and fingering changes: the pluck of an inharmonic string of any length would be like a chime at best.

8.7 Combining Modes

It took people a long time to get straight the idea of *superposition*—that you can add one mode to another and they just go on doing their thing independently. This is called *linear superposition*; we saw it in connection with waves passing right through each other (even though they have to interfere when they occupy the same space—it could not be any other way if they are to survive the "collision" with each other). You can run scenario A in the lab on Monday (for example, hit a metal plate on the left side), run scenario B on Tuesday (hit the same metal plate on the right side), collect the data, and on Wednesday add the data from Monday and Tuesday, using the moment of the strike as $t = 0$ in both cases. On Thursday, you can run both A and B at once, striking the plate on both sides simultaneously, and collect that data. On Friday, your analysis will reveal that the resulting sound and motion of the plate is the same, whether you really struck it in two places at once or added up the sound and motion from individual strikes on a computer. A good week's work, and you'll never forget the principle of linear superposition.[3]

The most general motion of the string is a superposition of the pure modes with amplitudes a_n and phases δ_n—that is,

$$y(x,t) = \sum_n a_n \sin(k_n x)\cos(2\pi f_n t + \delta_n), \tag{8.17}$$

[3]Many physical processes are nonlinear—that is, their response is not in proportion to the input. Fortunately, sound that we find not too loud is most often created and propagated by linear rules.