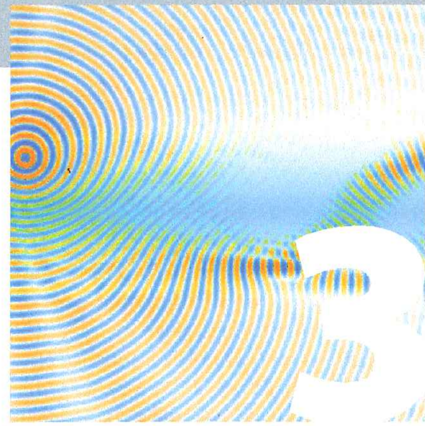

Sound and Sinusoids

The original sound analysis tool was the unaided human ear, and perhaps the occasional observation of a "sympathetic resonance" of an open string or tube in the presence of other sounds. However impressive, human hearing has finite frequency and dynamic ranges and, as we shall see, is easily fooled about the internal structure of sounds, musical or otherwise.

The beginnings of quantitative sound analysis were humble. In the second half of the nineteenth century, so-called Helmholtz resonators were constructed to aid the human ear in picking out individual components of musical sound. We will discuss them extensively in chapter 13. The situation has now completely changed. Given good microphones and a laptop, it is fair to say that where human hearing is concerned, there is little left to be desired in the way of sound analysis tools.

The pressure variation recorded at a point over a period of time (that is, sound) is an example of a *signal*. Quantifying and understanding the content of sound falls under the well-developed field of *signal analysis*.

There can be only one pressure at a given place and time. If a plot is made of a signal $f(t)$ versus time, a vertical line placed at any time must intersect the plot only once. As long as they obey this rule, signals can take any shape or form; the variety is infinite. One of the jobs of signal analysis is to decompose, or parse, signals into understandable components. Those components will themselves be simpler signals.

The question arises: Is there a "best" or uniquely fundamental type of signal, more perfect somehow than any other? It may seem an absurd abstraction or an idealization to ask whether there is a type of function more fundamental than any other. The question may make mathematicians very uneasy. There is, however, a physical answer to this question, because of the way objects almost universally vibrate or oscillate, especially for low amplitudes of oscillation. They vibrate *sinusoidally*, as one or a combination of sine functions, one for each frequency of vibration. In chapter 8,

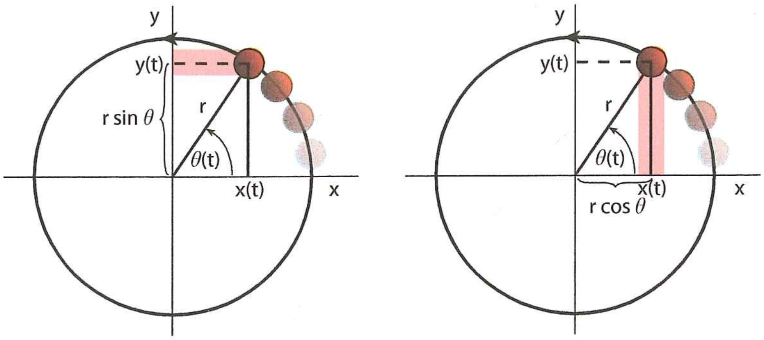

Figure 3.1
The red ball rotates counterclockwise at uniform speed and casts a shadow on the y axis, given by $y = r\sin\theta = r\sin(2\pi f t)$, and a shadow on the x axis, given by $x = r\cos\theta = r\cos(2\pi f t)$. The frequency f is the number of full revolutions per second, which could be any number, such as 2.257, 3, or 5000.

on string vibrations, the sinusoid will rule; here we anticipate sinusoidal supremacy and develop some implications.

3.1 The Atom of Sound

Without the sinusoid, it is impossible to fully understand sound and music. This is by no means completely obvious, and it wasn't fully appreciated until the early to mid-1800s. The establishment of the primacy of the sinusoid marks the beginning of the modern era of acoustics. More fundamental than any other type of vibration, sinusoidal vibration is truly the *atom of sound* from which all other vibrations and sounds are constructed.

Building a Sine Wave

Fortunately, a sinusoid is very simple and easily derived from a circle, as shown in figure. 3.1. Suppose a ball is rotating counterclockwise around the circle, at uniform speed. Its angle measured from the x axis, θ, is increasing at a steady rate, $\theta(t) = 2\pi f t$, where $2\pi f$ gives the rate of increase and t is the time. Note that the angle θ is measured in *radians*, and the full circle (360 degrees) is $2\pi = 6.28319\ldots$ radians. f is called the *frequency*. Starting at $t = 0$, the increase θ after one second ($t = 1$) is $2\pi f$. A full revolution is an angle of 2π, so after one second the ball will have made $2\pi f \cdot 1/(2\pi) = f$ revolutions—that is, the ball makes f revolutions per second. Note that f can be any number. Because 360 degrees is physically the same as 0 degrees, we usually agree to start over ofter one full revolution and keep the angle between 0 and 360 degrees.

Since it is going around at constant speed, the motion of the ball starts to repeat itself after exactly one revolution. The time it takes to go one revolution is called the period T, and the motion we say is periodic with period T. T is a solution to the equation $2\pi f T = 2\pi$, i.e., $T = 1/f$. The period is simply the inverse of the frequency.

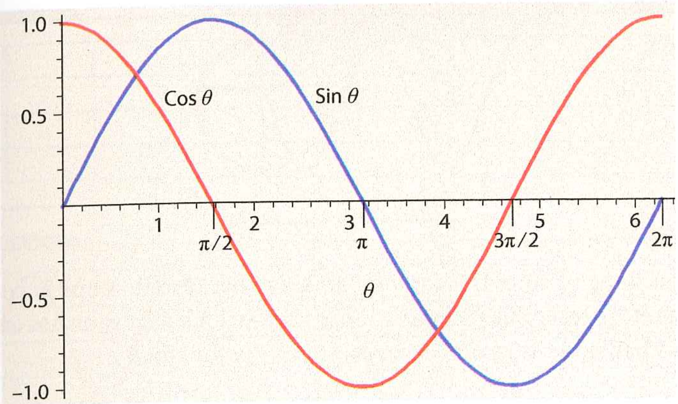

Figure 3.2
Sine and cosine are the same function except for a phase displacement of $\pi/2$.

It is appropriate to do a little dimensional analysis. Time is measured in seconds (s), so frequency must be measured in inverse seconds, or s^{-1}, in order for both sides of the equation $T = 1/f$ to have the same dimensions, which they must. When you think of frequency, you can mutter the words "per second" or "cycles per second" to yourself. One Hz is one cycle per second, and it has dimensions of s^{-1}.

The shadow, or projection, y of the ball on the y axis oscillates up and down as the ball circulates. Now comes the important statement: If the radius of the circle is r, the sine function is *defined* geometrically as

$$\sin\theta = y/r.$$

The cosine is similarly defined as

$$\cos\theta = x/r.$$

Since $x^2 + y^2 = r^2$ according to the Pythagorean theorem, we have proven the "trig identity" $\sin^2\theta + \cos^2\theta = 1$. If the angle θ is increasing steadily—that is, $\theta(t) = 2\pi ft$—then x is necessarily given by $x(t) = r\cos\theta(t) = r\cos(2\pi ft)$. This is the cosine function, and the oscillation along x is called sinusoidal. The plot of sine looks the same as cosine, except it is shifted by a phase $\pi/2$—that is, $\cos\theta(t) = \sin[\theta(t) + \pi/2]$ (see figure 3.2).

Some important qualitative aspects of sinusoidal functions are as follows:

- Sin x and $\cos x$ are simply phase-shifted versions of each other: $\sin(x + \pi/2) = \cos(x)$.
- Sin x and $\cos x$ top out at +1, and bottom out at −1.
- Sin x and $\cos x$ are perfectly periodic with period 2π.
- A plot of the slope of $\sin x$ is simply $\cos x$; the slope of $\cos x$ is simply $-\sin x$.

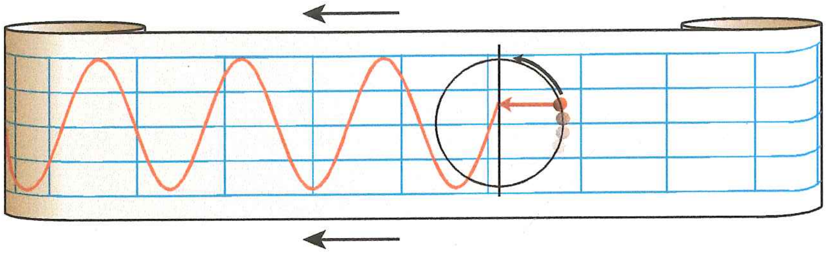

Figure 3.3
If we position a red stylus at the y shadow of the ball (the tip of the arrow, not the ball itself), it traces a sine wave as the chart paper moves and the ball rotates uniformly.

- The slope of $\sin x$ is 1 for very small x (because its slope is $\cos x$); it therefore resembles the function x itself, which also vanishes at $x = 0$ and has slope 1. In other words, $\sin x \sim x$ for small x.

Imagine a stylus with red ink whose tip (the red arrowhead in figure 3.3) is at the y position of the ball (not at the ball itself). As the ball rotates counterclockwise with uniform speed around the circle, we get the sinusoidal waveform plotted on the moving chart paper—that is, a plot of $y(t) = r\sin(2\pi f t)$ versus t, where t is marked out along the chart paper according to how fast it is moving.

We will often employ sines and cosines in describing sound and vibration. A pure tone, a sinusoid, or sinusoidal vibration of frequency f can be written as

$$y(t) = A\cos(2\pi f t + \phi), \tag{3.1}$$

where $y(t)$ is the signal, (for example, pressure variation), ϕ is a phase that displaces the cosine in time, and A is the *amplitude*.

3.2 Sinusoidal Vibration

Why is the sine wave as described earlier so fundamental to vibration and sound? The first hint of a preeminent role for the sinusoid comes from vibrations of physical objects. It is found that most objects resist displacement from their resting position, or distortion from their resting shape, by pushing back with a force proportional to the displacement or distortion. When released, the object accelerates toward its former resting position, responding to the force according to $F = ma$. (The rule governing the motion of the mass after it is let go is given by Newton's law, $F = ma$, where F is the force, m is the mass, and a is the acceleration, which is the rate of change of the velocity (v) of the displacement.) As it returns to its old resting position, however, the object finds itself moving fast and thus overshooting, causing the object to deform in the opposite sense. The object now resists deformation with a force in the opposite direction, causing a deceleration (which is just an acceleration in the opposite direction) in response. The motion stops (if there is no friction) only when the object has come to an equal but

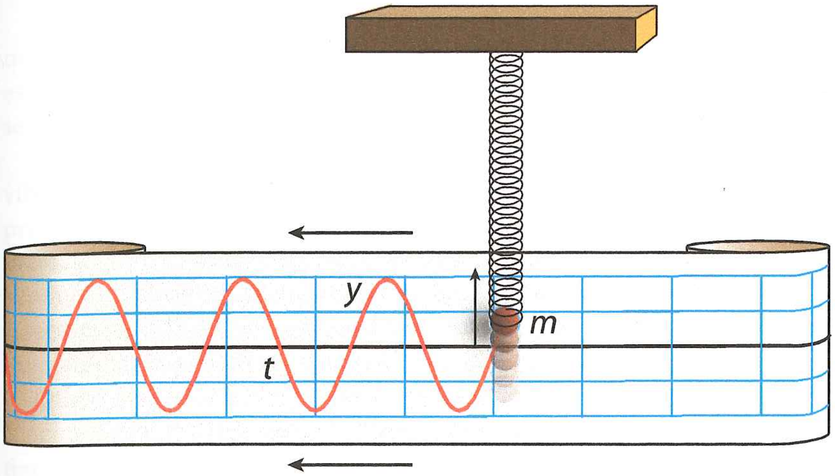

Figure 3.4
A mass *m* oscillating on a massless spring draws a sine wave on the chart paper as it moves by. This is called a *harmonic oscillator*; an often-used synonym for sinusoidal oscillation is *harmonic oscillation*.

opposite displacement. The continuing force again accelerates it toward its resting position, and again it overshoots, and so on. A periodic oscillation is set up.

The usual textbook example of this is a mass m hanging from an ideal spring (figure 3.4). If the displacement from rest position is y, an ideal spring resists with a force $f = -ky$, where k is the *force constant*. Suppose we "pluck" the mass by displacing it to a distance A from its resting position and then letting it go at time $t = 0$. The motion after that is given by

$$y(t) = A\cos(2\pi f t), \tag{3.2}$$

where the frequency is given by the important formula

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{3.3}$$

Thus *the motion of the mass is exactly sinusoidal*, and a plot of $y(t)$ versus t is a sinusoid. If we displace it more initially (make A bigger), it oscillates with larger amplitude A but its *period* $T = 1/f$ is the same. The sine function is yielding its first hints of its preeminent role in vibration and sound. The sinusoid applies to all systems for which restoring force is proportional to displacement. This force rule applies to all sorts of random objects when they are displaced just a little.

The Velocity

Just when the ball reaches its maximum or minimum excursion, the speed is zero. Conversely, just as the ball passes through zero displacement—that is, $y = 0$—it is moving the fastest. The velocity of the ball is the rate of

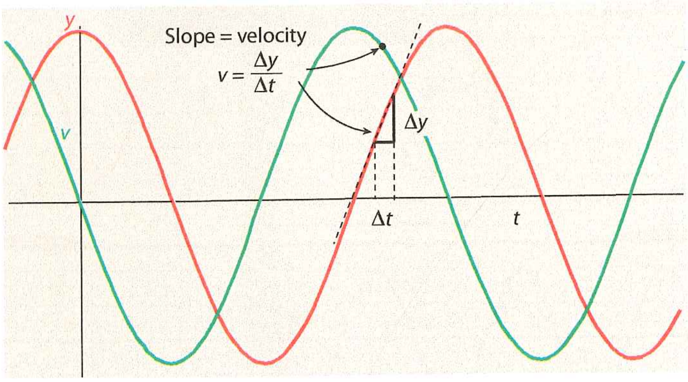

Figure 3.5

The red sinusoidal trace of the ball is seen along with a plot of the velocity (green) and the construction of the velocity as the slope $v = \Delta y/\Delta t$ of the plot of position versus time. The velocity $v(t)$ is 90 degrees out of phase with the displacement $y(t)$.

change of its position—in other words, the slope of the curve $y(t)$:

$$v(t) = \frac{\Delta y}{\Delta t}. \tag{3.4}$$

The graphical construction of the velocity is shown in figure 3.5. The green velocity curve is displaced by a quarter of a period, or 90 degrees, from the red displacement curve. We say the velocity is "90 degrees out of phase" with the displacement.

The Tuning Fork

The tuning fork seems a rather humble device that does but one thing well—vibrate at a definite frequency and make a pure but almost barren tone at that frequency. The tuning fork has played a leading role in music and sound ever since its invention in 1711 by John Shore, who, like his father before him, was sergeant trumpeter to the court of King George I. Shore was a friend of George Frederic Handel, who wrote parts specifically for him, as did Henry Purcell. He was one of the few trumpeters who could play Purcell's notoriously difficult passages. He carried a tuning fork around in his pocket.

The tuning fork seems like a simple invention, but there are not many metal objects you can hold while they ring a pure tone for a minute, which a good tuning fork will do. Most objects make only an indistinct pitch or a tone with many frequencies in it, unsuitable for a frequency standard. The tuning fork has the rare distinction of barely shaking a large part of itself—the stem—as it vibrates in the mode easily excited by striking the prongs (see figure 3.6). Thus there is little tendency for soft fingers or the support to sap the vibrational energy of the prongs. If the fork is placed on a resonator (more about resonators in chapter 13), the sound is amplified—in fact, sent out by a completely different mechanism. The resonator box receives tiny residual vibrations of the stem and turns them into sound much more efficiently than the stem can; it also captures the *near field*, as it is called, of the tuning fork and reradiates it. (See box 20.1 and nearby sections.)

Most objects are very poor at converting vibrational energy sound. The vibrations damp out mainly due to other sources of friction, not to sound production. Inducing such objects to produce sound more efficiently barely increases the overall damping; in other words, that part due to sound production remains small compared to other sources of damping. This is a seemingly modest observation but is actually a key to many phenomena involving sound production.

Other modes of vibration of the tuning fork exist, as with all macroscopic objects, having higher frequencies than the vibration described in figure 3.6. But if these higher modes get excited when the fork is struck, most tend to rapidly damp out, especially when the base is being held, leaving only the lowest tone. Later (see section 15.7), we discuss the recently discovered *Belleplates*, flat metal sheets with a special shape and a tab for holding, which also ring for many seconds.

Before the tuning fork was used as a frequency standard, pitch could vary by as much as four semitones (two whole notes of the Western musical scale) from one orchestra to the next, or even from one instrument to the next. But even the tuning fork (and the pitchpipe) didn't prevent the gradual creep upward of concert pitch in Europe, beginning around the seventeenth century. Vocalists tried to rein in this rise, citing strain on their voices.

In spite of the popularity of the tuning fork as an instrument useful to orchestras and choruses, no one knew how they really worked. Ernst Chladni, whom we will meet again in connection with vibrating metal plates (see sections 15.3 and 15.4) began the tuning fork's transformation into a scientific instrument 90 or so years after its invention. He found that the tines vibrate in synchrony, both moving toward the middle and then both moving out, pivoting from the U-shaped base, where there is very little movement (see figure 3.6).

The tines of the tuning fork are responsible for both its mass and its springiness. This is typical of most objects: the "massiness" and the "springiness" are both distributed throughout the object. For small displacements of the forks, the restoring force is proportional to the displacement. Therefore, we expect the vibration of the fork to be sinusoidal (figure 3.7), provided only one mode of oscillation is excited. One of the advantages of the tuning fork design is that it is very easy to excite the intended mode with both tines vibrating symmetrically by striking the fork against an object. We shall learn much more about multiple vibrational modes, their periods, and their combinations when studying the bead system in the next chapter, but they are introduced in this chapter in connection with the double tuning fork.

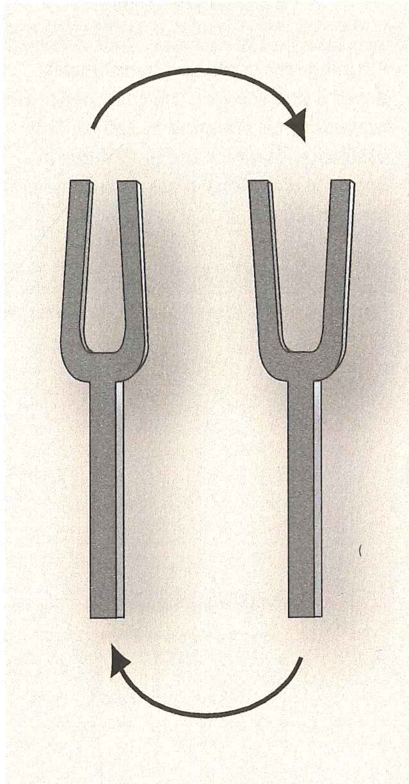

Figure 3.6

Fundamental vibration mode of the prongs, or tines, of a tuning fork. A tuning fork is typical of vibrating objects, in that the "mass" and "spring" are not separated. It has a prominent symmetric oscillation of the tines at its design frequency. The tines push air back and forth periodically, which causes pressure waves to form and radiate away.

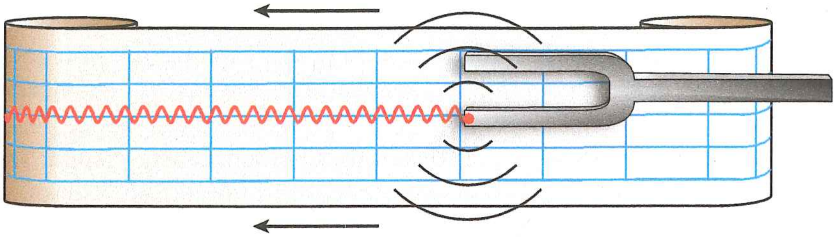

Figure 3.7
A tuning fork oscillating harmonically draws a sine wave on the chart paper as it moves by; it is vibrating at 220 Hz. This oscillation, in which the tines move in opposite directions, gives an audible pure tone at 220 Hz.

The Sound of a Sinusoid

The tuning fork in its lowest, most easily excited mode of vibration emits just one sinusoidal tone at a single frequency. This is a "pure" or "simple" tone—the atom of sound. A sinusoid is perfectly periodic, with the period given by the inverse of the frequency. Most sound sources generate complex tones consisting of many simultaneous sinusoids, even if they are periodic. It was by no means obvious in the nineteenth century that the only correlate to a pure tone is the sinusoid. This is the essence of Ohm's law, which will be taken up in earnest in the chapters on pitch (chapter 23) and timbre (chapter 24).

The barren sound of a sinusoidal pure tone is described in various ways: plain, no color, clinical, and so on. Paul Falstad's *Fourier* applet readily generates single sinusoidal tones.

3.3 The Pendulum

I took two balls, one of lead and one of cork, the former more than a hundred times heavier than the latter, and suspended them by means of two equal fine threads, each four or five cubits long. Pulling each ball aside from the perpendicular, I let them go at the same instant, and they, falling along the circumferences of circles having these equal strings for semi-diameters, passed beyond the perpendicular and returned along the same path. This free vibration repeated a hundred times showed clearly that the heavy body maintains so nearly the period of the light body that neither in a hundred swings nor even in a thousand will the former anticipate the latter by as much as a single moment, so perfectly do they keep in step.
—Galileo, *Two New Sciences*, 1638

The pendulum is the first and still perhaps the most important paradigm for sinusoidal vibration. It is also the source of another synonym for sinusoidal vibration: *pendular vibration.*

Galileo remarks that the period of the pendulum does not depend on the mass of the bob nor the amplitude of its swing, unless it exceeds

90 degrees. In fact, the period of the pendulum is very nearly but not exactly independent of the amplitude for small amplitudes, but by 90 degrees and more it starts to increase significantly. For small amplitudes, the period T is given by

$$T = 2\pi\sqrt{\frac{\ell}{g}}, \tag{3.5}$$

where ℓ is the length of the pendulum in meters, and g, the gravitational constant, is on average over the earth's surface $g = 9.80665\ \mathrm{m/s^2}$, but the actual number varies by as much as half a percent from place to place on earth. This is a fairly large variation—for example, a pendulum clock would not keep good time without a correction in England if it kept good time in Italy.

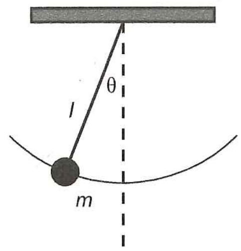

Figure 3.8
Simple pendulum of length ℓ and mass m.

Try experimenting with a pendulum consisting of a thread and a set of keys. You can check equation 3.5 since it involves only the length of your thread and a known constant. If you want to be precise, you should try to estimate the position of the center of gravity of your keys when calculating the length ℓ, and of course use meters to measure length. The mass of your keys does not come into the formula, nor does the amplitude of the swing if it is small; the latter is an approximation good for small amplitude (see figure 3.8).

The motion of the pendulum is given by

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right), \tag{3.6}$$

where θ_0 is the initial displacement of the pendulum at its maximum angle. If we run our chart paper recorder, as shown in figure 3.9, the track of the pendulum bob (for small amplitude) is very nearly sinusoidal.

We have already accumulated three systems that vibrate sinusoidally: the mass and spring, the tuning fork, and the pendulum. A key point, not yet obvious from what we have said so far, is that most objects can vibrate in many ways, even thousands or millions of ways, and yet each of these individual ways is itself a sinusoidal oscillation, in that any point on the object is moving up and down, or side to side, and so on, sinusoidally. Hardly a more important principle can be expressed in all of the subject of acoustics, but we shall have plenty of opportunities to point it out again—for example, in the next section. The system considered in chapter 8—namely, a set of beads that are held on filament under tension—will help to clarify this point.

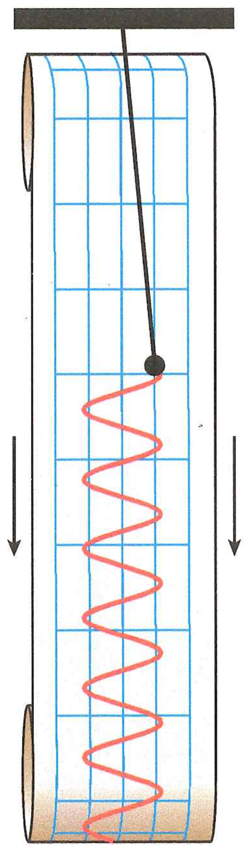

Figure 3.9
A swinging pendulum bob traces out a sinusoid on moving graph paper.

3.4
The Double Tuning Fork

The sinusoid reigns supreme not because it describes the vibration of masses and springs, tuning forks, and pendula, but rather because *any*

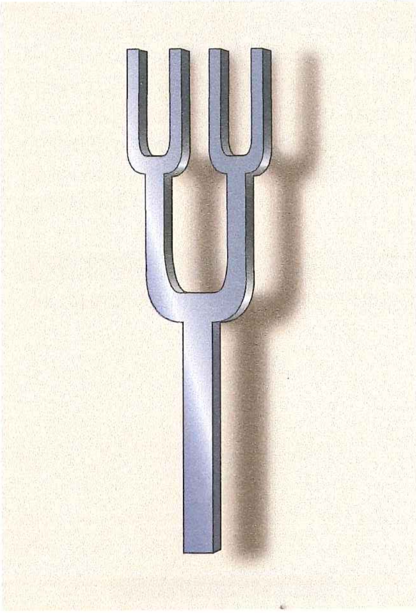

Figure 3.10

A double tuning fork. The smaller tuning forks are carried back and forth as a whole as the large tines oscillate. In addition, the smaller tines oscillate at a higher frequency without significantly affecting the large tines.

sound (or indeed any signal) and any vibration can be decomposed into a sum of different sinusoids. A single sinusoid oscillating at a fixed frequency is the source of a pure tone. This is what makes the sinusoid the atom of sound. Complex molecules, just like complex tones, can be broken down into their constituent atoms. Complex tones (molecular sound so to speak) such as that produced by a trumpet can be decomposed in terms of simple sinusoidal tones (atomic sound). We have not yet established these concepts, but the first hints are emerging.

We can ease into these concepts with the aid of a new kind of tuning fork. Perhaps no one has ever constructed a double tuning fork of the type shown in figure 3.10. But it could be done, and as a thought experiment it is very instructive. It will lead us into the world of periodic and nonperiodic functions that can be decomposed into sine waves.

Our double tuning fork has tines that are themselves tuning forks. To understand how this would work, we can first ignore the smaller tuning forks and suppose that the smaller tines of the secondary forks are not moving relative to each other, making the smaller forks act as a solid mass. This is actually a possible mode of vibration by which the complex tuning forks could oscillate, and except for the strange shape of the forks not much has changed from the original tuning forks we considered already. We hold the main base steady and horizontal, labeling the vertical position of the corner of one of the small forks by the coordinate $y(t)$. Then, in the mode just described,

$$y(t) = a_1 \cos(2\pi f_1 t), \tag{3.7}$$

where f_1 is the frequency of the vibration that oscillates the tines as a whole, and a_1 is the initial displacement.

Since tuning forks don't shake their base very much, another possible mode of vibration is for the smaller forks to vibrate without setting into motion the lower frequency mode in which the smaller forks are carried back and forth as a whole. This motion also displaces the position of the smaller tines and is described by the equation

$$y(t) = a_2 \cos(2\pi f_2 t), \tag{3.8}$$

where f_2 is the frequency of the vibration that oscillates the smaller tines as if they were not attached to something larger, and a_2 is the initial displacement in this mode.

It would be quite possible to have the double fork vibrate at three different frequencies, by leaving the mass of the left-and righthand smaller forks equal, but reshaping one of the smaller forks so that it vibrates at a different frequency than the other small fork. Even though there are in reality myriad other ways this complex object can vibrate, the three modes in which the left small fork opens and closes (which we call mode L), the right small fork opens and closes (which we call mode R), and both small forks are shaken as a whole (which we call mode B), are by

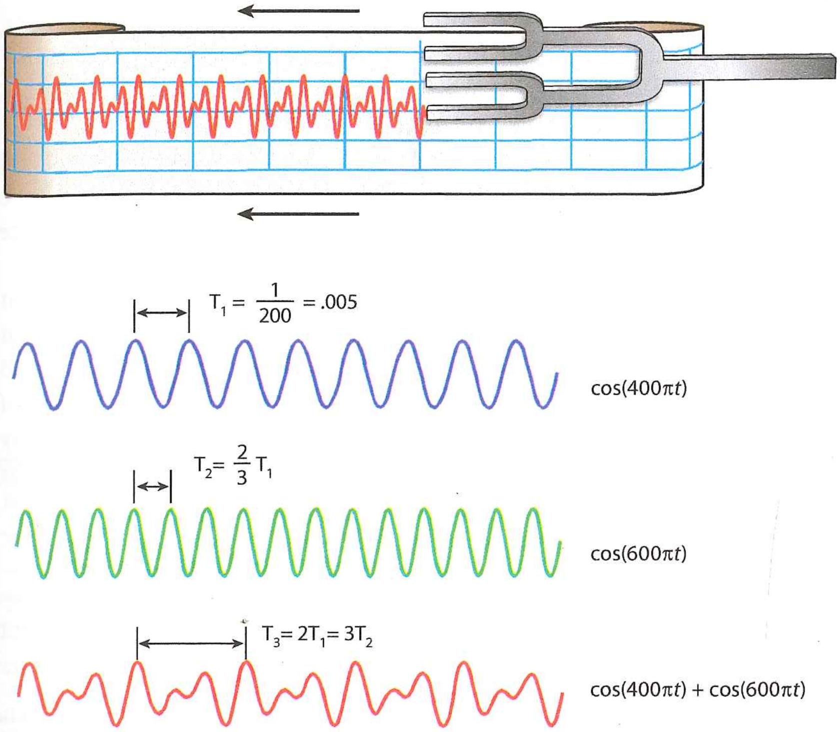

Figure 3.11
Trace of the stylus of a double tuning fork with two modes excited. In the case shown, the smaller fork is vibrating faster than the larger fork by a factor of 3/2. Notice that the period of the sum of the two sinusoids, $\cos(2\pi \cdot 200t) + \cos(2\pi \cdot 300t)$, is double that of $\cos(2\pi \cdot 200t)$—that is, 0.01 second. This fact will play a large role later, when we consider the perception of pitch.

far the most important and easy to excite if the fork is being held at its base.

Last, there is no reason why all three modes could not be excited at once. The lower frequency mode B, carrying the smaller forks back and forth as a whole, is scarcely affected by whether the smaller forks are vibrating a little. Similarly, the smaller tines barely register whether their base is swaying back and forth more slowly, since the vibration of the tines is very weakly coupled to the base. For two modes excited, the total displacement of one of the smaller tines is now a sum of the overall displacement of the small tuning fork, plus the displacement of the small tines relative to its base:

$$y(t) = a_1 \cos(2\pi f_1 t) + a_2 \cos(2\pi f_2 t). \tag{3.9}$$

That is, the position of the smaller tines is first displaced by the lower frequency mode and then figured from there by adding the displacement of the smaller fork's higher frequency mode, making $y(t)$ a sum of two sinusoids of different frequencies (see figure 3.11).

What can result from this? One important question is whether the two frequencies are commensurate. By that we mean, is some integer multiple of one of the frequencies exactly equal to an integer multiple of another? This question will be more meaningful, in music and in practice, if the integers are small. In other words, we are not usually interested in whether $2057 f_1 = 1317 f_2$, but we *are* interested if $2 f_1 = f_2$ or $3 f_1 = 2 f_2$.

If the frequencies of the two vibrations are commensurate—for example, $3f_1 = 2f_2$, or more generally $mf_1 = nf_2$ for small integers m and n—the path $y(t)$ traced out by the fork will be periodic, with a period. This is made clear in the following. If the two frequencies are commensurate but only with large integers involved, such as $m = 16$ and $n = 17$, the trace will still be periodic, but with a longer period. Last, the two frequencies could have a ratio that is an irrational number such as the $\sqrt{2}$—in that case, the trace never exactly repeats any pattern.

It is important to note that by striking the double fork in the right way, we can get either mode L or R vibrating independently and without exciting mode B. Likewise, it is possible to get the lowest frequency mode B activated without activating either of the smaller forks. In this way, each of the frequencies may be heard as a pure tone, or they may be heard in any combination. That combination could be musical or not, and pleasant or not, depending on the frequencies to which the forks are tuned.

3.5 Microscopes for Vibration

It is possible to greatly amplify the small excursions of vibrating objects by attaching a mirror to them and then shining a beam of light onto the mirror. Small differences in the angle of bounce during a vibration can be magnified into big excursions on a wall some distance away. A deflection of only 1/10 of a degree is enough to cause the spot of reflected light to move about 1 cm on a wall 5 m away from the mirror. In figure 3.13, a narrow light beam (a laser beam would be ideal today) strikes first one and then another mirror attached to its own tuning fork. The forks are slightly mistuned, so that the mirrors at first vibrate in phase, then out of phase, and so on. After bouncing off both mirrors, the light beam falls on a wall. The direction of the beam is sensitive to slight angle changes of the mirror as it rides back and forth on the tuning fork tines. If the second mirror is deftly rotated, the projection on the screen traces out the beat pattern shown in this figure. Jules Antoine Lissajous's (figure 3.12) scheme uses this principle and is a kind of microscope, magnifying the oscillation of the vibrating prong.

Figure 3.12
Jules Antoine Lissajous. Courtesy David Gerard.

Hermann von Helmholtz modified one of Lissajous's designs—a microscope altered so that the small objective lens placed just above the sample is also vibrating, riding on the prong of a tuning fork (figure 3.14). Helmholtz added an electromagnetic drive for the tuning fork, keeping the prongs vibrating steadily with a controllable amplitude. Thus was born the *vibration microscope*, the instrument Helmholtz used to demonstrate the kink wave traveling around a bowed violin string (see section 18.1), a fundamental discovery in musical acoustics. The optical axis (direction of view) of the lens is perpendicular to the plane of vibration of the prong. The lens is part of an otherwise standard compound microscope with the usual

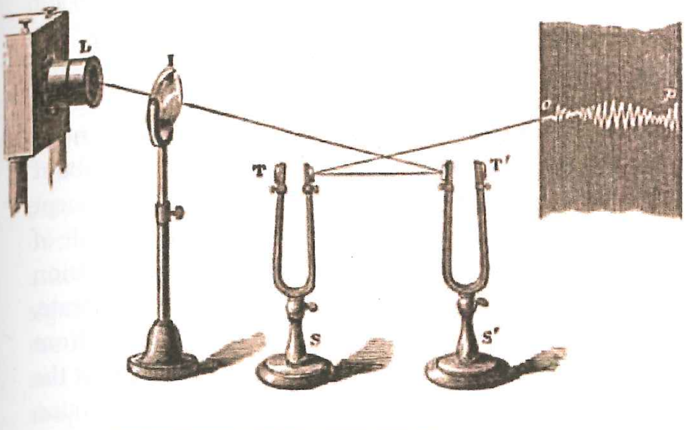

Figure 3.13
Lissajous's apparatus for visualizing beats.

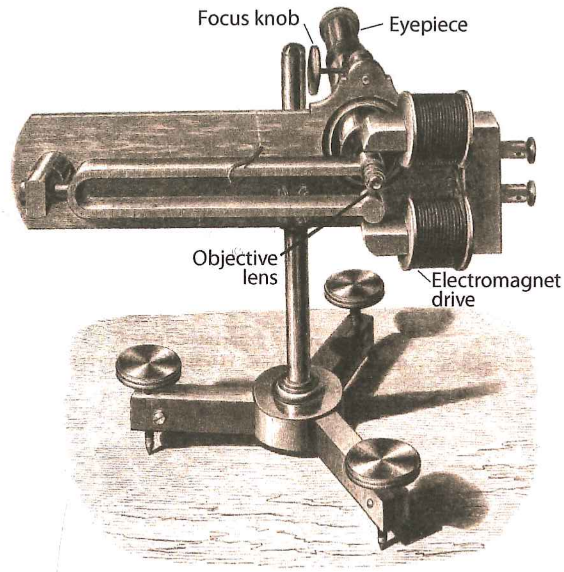

Figure 3.14
Helmholtz's version of the vibration microscope, this one made by Rudolph Koenig, which Helmholtz used to discover the kink wave traveling along a bowed violin string.

eyepiece held fixed. If the prong is vibrating and a stationary small white dot is viewed, it appears to oscillate, due to the oscillation of the objective lens.

Suppose a small white dot is painted on another body vibrating at right angles to the vibration of the fork and to the axis of the microscope. Viewing this through the microscope, we will see the result of both the dot vibration and the lens vibration. The lens, attached to the tine, executes a simple harmonic oscillation of known frequency. Now a question arises: what is the track taken by the small white dot when viewed through the microscope? It is clear that if the period of the second vibration is the same as that of the prong, the dot will trace a closed curve. If the second vibration is also harmonic, this curve will be an ellipse. If the period is the same as the prong but the second vibration is not harmonic, the curve traced will be more complex. When analyzed, however, the curve obtained will yield the secret of the motion of the second vibration. If the second vibration is twice or half the period of the tine, it is still clear that a closed curve will result, but the curve traced will not be as simple and will also depend on the relative phase of the two vibrations. The same holds true if the ratio of frequencies is any small integer ratio such as 3:4.

If the ratio cannot be expressed so simply, the curve traced out appears to be slowly rotating through different shapes it attains with various relative phases of a nearby integer ratio. For example, if the ratio is 3.073317:4, the curve will very nearly trace out all the possible phases of the exact 3:4 case. These considerations lead us down the path of number theory, rational approximations to real numbers, and so on, which is a beautiful but very deep subject. We shall have to break off here, with the exception of presenting some of the Lissajous patterns seen for various types of motion (figure 3.15).

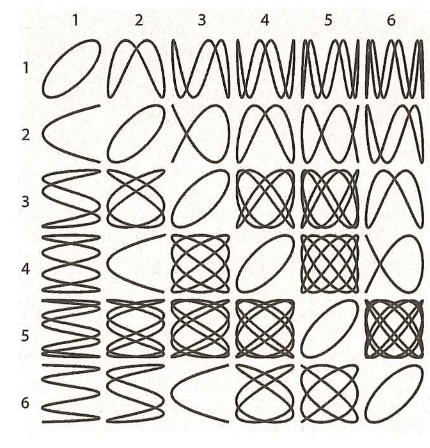

Figure 3.15
Array of Lissajous figures for two tuning forks, one carrying the objective lens, the other a white dot whose tracings are shown here. The ratio of frequencies of the forks is $n{:}m$, $n = 1, \ldots, 6$, $m = 1, \ldots, 6$ for a given relative phase of the two forks.