Earth Deformation
Homework 1

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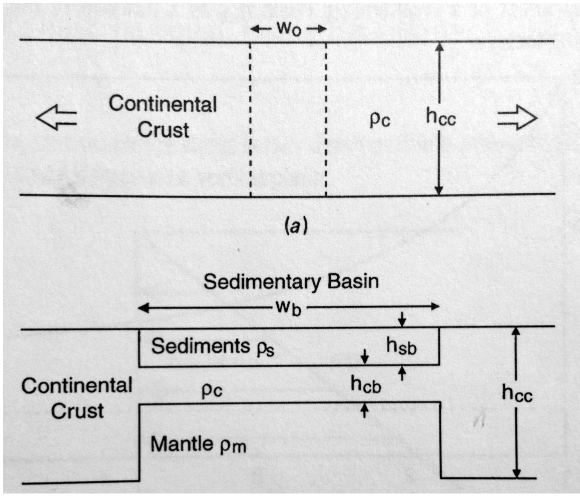
Problem 1 (T+S Problem 2-5)

We assume the setup of Figure 2-4 from Turcotte and Schubert:



We are given the following values:

$$h_{cc} = 35 \text{ km}$$
$$h_{sb} = 7 \text{ km}$$

$$\rho_m = 3300 \text{ kg/m}^3$$
$$\rho_{cc} = 2700 \text{ kg/m}^3$$
$$\rho_s = 2450 \text{ kg/m}^3$$

We need to solve equation (2-10):

$$h_{sb} = h_{cc} \left(\frac{\rho_m - \rho_{cc}}{\rho_m - \rho_s} \right) \left(1 - \frac{1}{\alpha} \right)$$

Plugging in the values above we get:

$$\boxed{\alpha \equiv \frac{w_b}{w_0} \approx 1.4}$$

Problem 2 (Rotational symmetry of a crystal)

Let us take some matrix A:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Matrix of a 90-degree counter-clockwise rotation about x:

$$R_x(90) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Matrix of a 90-degree counter-clockwise rotation about y:

$$R_y(90) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Matrix of a 90-degree counter-clockwise rotation about z:

$$R_z(90) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix A rotated 90 degrees about x:

$$A_{\text{rot.x}} = R_x A R_x^{-1} = \begin{pmatrix} a & -c & b \\ -g & i & -h \\ d & -f & e \end{pmatrix}$$

The matrix A rotated 90 degrees about y:

$$A_{\text{rot.y}} = R_y A R_y^{-1} = \begin{pmatrix} i & h & -g \\ f & e & -d \\ -c & -b & a \end{pmatrix}$$

The matrix A rotated 90 degrees about z:

$$A_{\text{rot.z}} = R_z A R_z^{-1} = \begin{pmatrix} e & -d & -f \\ -b & a & c \\ -h & g & i \end{pmatrix}$$

The problem states that we must have:

$$A_{\text{rot.x}} = A_{\text{rot.y}} = A_{\text{rot.z}} = A$$

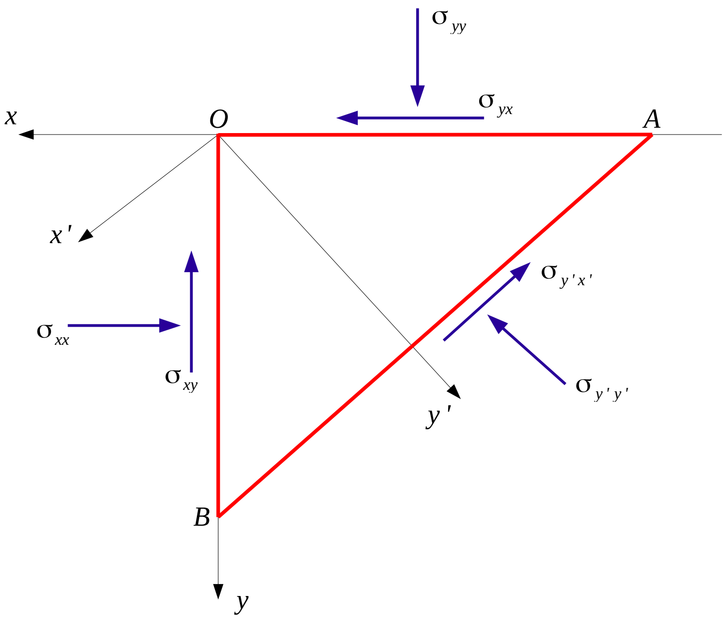
Thus we can conclude that all of the off-diagonal components must be zero and that $a = e = i$ such that the matrix A must be of the form:

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

We therefore conclude that the matrix A has only *one* independent component!

Problem 3 (T+S Problem 2-13)

We can construct the following diagram:



Force balance in the $\hat{x}$ direction tells us that:

$$\sigma_{yx}\text{OA} - \sigma_{xx}\text{OB} - \sigma_{y'x'}\cos(\theta)\text{AB} + \sigma_{y'y'}\sin(\theta)\text{AB} = 0$$

We can divide both sides by AB and rearrange:

$$\sigma_{y'x'}\cos\theta - \sigma_{y'y'}\sin\theta = \sigma_{yx}\cos\theta - \sigma_{xx}\sin\theta$$

Now multiply both sides by $\sin\theta$:

$$(\star) \qquad \sigma_{y'x'}\sin\theta\cos\theta - \sigma_{y'y'}\sin^2\theta = \sigma_{yx}\sin\theta\cos\theta - \sigma_{xx}\sin^2\theta$$

Force balance in the $\hat{y}$ direction tells us that:

$$\sigma_{yy}\text{OA} - \sigma_{xy}\text{OB} - \sigma_{y'x'}\sin(\theta)\text{AB} - \sigma_{y'y'}\cos(\theta)\text{AB} = 0$$

We can divide both sides by AB and rearrange:

$$\sigma_{y'x'}\sin\theta + \sigma_{y'y'}\cos\theta = \sigma_{yy}\cos\theta - \sigma_{xy}\sin\theta$$

Now multiply both sides by $\cos\theta$:

$$(\star\star) \qquad \sigma_{y'x'}\sin\theta\cos\theta + \sigma_{y'y'}\cos^2\theta = \sigma_{yy}\cos^2\theta - \sigma_{xy}\sin\theta\cos\theta$$

Now use the symmetry of the stress tensor to say that $\sigma_{xy} = \sigma_{yx}$ and subtract $(\star)$ from $(\star\star)$:

$$\sigma_{y'y'}(\sin^2\theta + \cos^2\theta) = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta$$

Now recall that $\sin^2\theta + \cos^2\theta = 1$ and $2\sin\theta\cos\theta = \sin 2\theta$. So we see that:

$$\boxed{\sigma_{y'y'} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - \sigma_{xy}\sin 2\theta}$$

Problem 4 (Stress tensor)

We are given the following stress tensor:

$$\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

We can calculate the pressure p:

$$p \equiv \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = \frac{1+1+2}{3} = \frac{4}{3}$$

So we know the isotropic part of σ:

$$\sigma_{\text{iso}} = \begin{pmatrix} \frac{4}{3} & 0 & 0 \\ 0 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

And the deviatoric part of σ:

$$\sigma_{\text{dev}} = \sigma - \sigma_{\text{iso}} = \begin{pmatrix} -\frac{1}{3} & 1 & 0 \\ 1 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}$$

The eigenvalues of σ are:

$$\sigma_{\text{I}} = 2$$
$$\sigma_{\text{II}} = 2$$
$$\sigma_{\text{III}} = 0$$

And the corresponding eigenvectors are:

$$\text{I} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

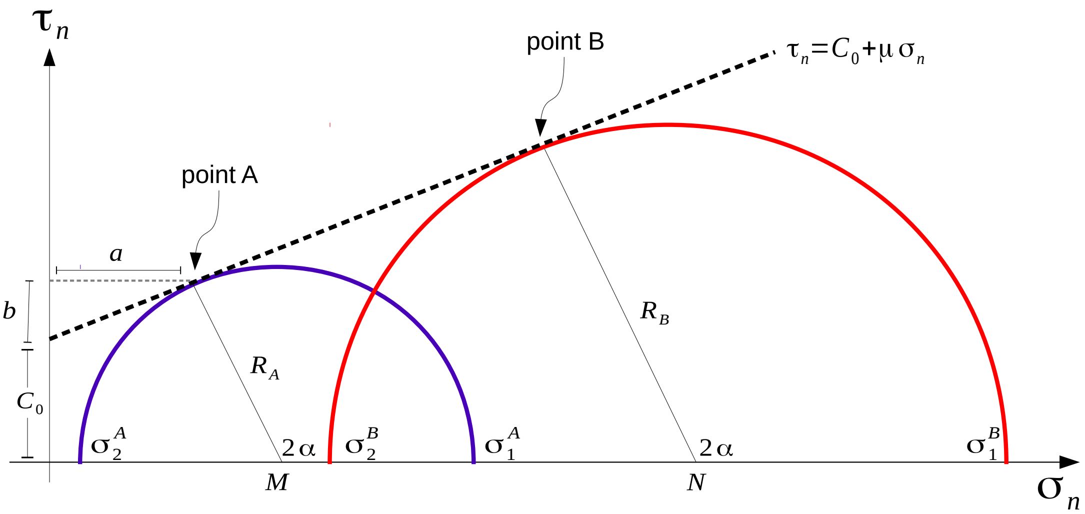
$$\text{II} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{III} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

It is easy to see that these three eigenvectors are all mutually orthogonal (the dot product of any two equals zero) — they form an "orthonormal basis."

Problem 5 (Mohr circle)

We can construct the following schematic diagram:



We can calculate the slope μ from point A to point B as a function of the angle 2α:

$$(\star) \qquad \mu = \left(\frac{\Delta\tau_n}{\Delta\sigma_n}\right)_{A\to B} = \frac{(R_B - R_A)\sin 2\alpha}{(\sigma_2^B - \sigma_2^A) + (R_B - R_A)(1 + \cos 2\alpha)}$$

We also know that the line perpendicular to $\tau_n = C_0 + \mu\sigma_n$ drawn from M to A (*or* from N to B) has a slope of $-1/\mu$:

$$(\star\star) \qquad -\frac{1}{\mu} = \left(\frac{\Delta\tau_n}{\Delta\sigma_n}\right)_{M\to A} = \frac{R_A \sin 2\alpha}{(\sigma_2^A + R_A + R_A\cos 2\alpha) - (\sigma_2^A + R_A)} = \tan 2\alpha$$

Recall that we know the following values:

$$\sigma_1^A = 1100 \text{ MPa}$$
$$\sigma_2^A = 20\text{MPa}$$

$$\sigma_1^B = 1700 \text{ MPa}$$
$$\sigma_2^B = 40 \text{ MPa}$$

$$R_A = \frac{\sigma_1^A - \sigma_2^A}{2} = 540 \text{ MPa}$$
$$R_B = \frac{\sigma_1^B - \sigma_2^B}{2} = 830 \text{ MPa}$$

We can now solve $(\star)$ and $(\star\star)$ for μ and α:

$$\boxed{\mu \approx 2.65}$$

$$\boxed{\alpha \approx 1.39 \text{ radians} \approx 79.65 \text{ degrees}}$$

We can use some geometry to solve for C_0 (see the labels on the diagram):

$$\cos(2\alpha - \pi/2) = \frac{a}{h}$$

We know a so we can further say:

$$h = \frac{\sigma_2^A + R_A + R_A \cos 2\alpha}{\cos(2\alpha - \pi/2)} \approx 155 \text{ MPa}$$

Now:

$$b = h \cos 2\alpha \approx 145 \text{ MPa}$$

So:

$$\boxed{C_0 \approx 45.64 \text{ MPa}}$$

Now we suspect that a 3$^{\text{rd}}$ sample had a pre-existing crack. Then $C_0 = 0$ and we can calculate the two points where the failure criterion curve intersects the Mohr circle. For the 1$^{\text{st}}$ test: $72.6° < \alpha < 86.7°$. For the 2$^{\text{nd}}$ test: $74.0° < \alpha < 85.3°$. **N.B.** You can solve the whole problem by plotting Test 1 and Test 2 on graph paper!

Problem 6 (Strain accumulation at the San Andreas Fault)

We are given the deformation gradient tensor:

$$D = \begin{pmatrix} 0.15 & 0.24 \\ 0.00 & -0.15 \end{pmatrix}$$

We can decompose the deformation gradient tensor into a symmetric tensor and an antisymmetric tensor:

$$D = \begin{pmatrix} 0.15 & 0.12 \\ 0.12 & -0.15 \end{pmatrix} + \begin{pmatrix} 0.00 & 0.12 \\ -0.12 & 0.00 \end{pmatrix}$$

We know that the San Andreas Fault trends N65°W so let us rotate the strain tensor counter-clockwise into that coordinate system:

$$\epsilon^* = \begin{pmatrix} \cos 65° & \sin 65° \\ -\sin 65° & \cos 65° \end{pmatrix} \begin{pmatrix} 0.15 & 0.12 \\ 0.12 & -0.15 \end{pmatrix} \begin{pmatrix} \cos 65° & \sin 65° \\ -\sin 65° & \cos 65° \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -0.004 & -0.192 \\ -0.192 & 0.004 \end{pmatrix}$$

We see that the fault-shear strains (off-diagonal components) are rather large compared to the fault-normal strains (diagonal components). This is exactly what we expect for a strike-slip fault such as the San Andreas! The dilatation Δ equals the trace (sum of the diagonal elements) of ϵ. Thus we can see $\Delta = 0$.

Problem 7 (T+S Problem 3-19)

Equation (3-132) tells us the functional form $w(x)$ of the plate deflection:

$$w(x) = w_0 e^{-x/\alpha} \left(\cos \frac{x}{\alpha} + \sin \frac{x}{\alpha} \right)$$

To get to the bending moment M we need to calculate the second derivative of $w(x)$:

$$\frac{d^2w}{dx^2} = \frac{2w_0 e^{-x/\alpha}}{\alpha^2} \left(\sin \frac{x}{\alpha} - \cos \frac{x}{\alpha} \right)$$

The bending moment M is proportional to the second derivative of $w(x)$:

$$M = -D \frac{d^2w}{dx^2}$$

We want to calculate the maximum value of M so we must take a derivative and set it equal to zero:

$$\frac{dM}{dx} = \frac{4D}{\alpha^3} e^{-x/\alpha} \cos \frac{x}{\alpha} = 0$$

The above is true for:

$$\boxed{x_{\max} = \pm \frac{\pi\alpha}{2}}$$

Now we can plug back in for M_m and get equation (3-138):

$$\boxed{M_{\max} \approx -0.416 \frac{Dw_0}{\alpha^2}}$$

We can use the values given to estimate the maximum bending moment in the lithosphere:

$$M_{\max} \approx -1.6 \times 10^{17} \text{ N}$$

The maximum bending (fiber) stress $\sigma_{xx}^{\max}$ is then given by equation (3-86):

$$\boxed{\sigma_{xx}^{\max} = \pm \frac{6M}{h^2} \approx 8.1 \times 10^8 \text{ N/m}^2}$$

The + corresponds to the tensile stress at the top of the plate and the − corresponds to the compressive stress at the bottom of the plate.

Problem 8 (T+S Problem 3-22)

We are told that the Amazon River Basin has a width $w = 400$ km. We are to model the basin as an elastic plate subject to a central line load (see Figures 3-29 and 3-30). We can use equation (3-135) to solve for the flexural parameter α:

$$x_b = \pi\alpha$$

$$\frac{400 \text{ km}}{2} = \pi\alpha$$

$$\alpha \approx 64 \text{ km}$$

We know that $(\rho_m - \rho_s) = 700$ kg/m^3 so we can solve for the flexural rigidity D:

$$\alpha = \left[\frac{4D}{(\rho_m - \rho_s)g}\right]^{1/4}$$

$$D \approx 2.9 \times 10^{22} \text{ Nm}$$

Now we can solve for the thickness T_e of the elastic lithosphere:

$$D = \frac{ET_e^3}{12(1-\nu^2)}$$

$$\boxed{T_e \approx 17 \text{ km}}$$

Note the rather small value of T_e here.

Problem 9 (Dabbahu laccolith)

Start from the flexure equation:

$$D\frac{d^4w}{dx^4} = q(x) - P\frac{d^w}{dx^2}$$

Note that $P = 0$ here. We can separate variables and integrate the equation four times to get:

$$w(x) = \frac{qx^4}{24D} + \alpha x^3 + \beta x^2 + \gamma x + \delta$$

The Greek letters are constants to be determined by four boundary conditions:

$$w(\pm L/2) = 0$$

$$\left.\frac{dw}{dx}\right|_{x=\pm\frac{L}{2}} = 0$$

The first two BCs tell us that $\alpha = \gamma = 0$. The second two BCs tell us:

$$\beta = \frac{-qL^2}{48D}$$

$$\delta = \frac{qL^4}{384D}$$

So we get:

$$w(x) = \frac{qx^4}{24D} + \frac{qL^2x^2}{48D} + \frac{qL^4}{384D}$$

If we define:

$$w_0 \equiv \frac{qL^4}{384D}$$

Then we can rewrite $w(x)$ as:

$$w(x) = w_0 \left(1 - \frac{8}{L^2}x^2 + \frac{16}{L^4}x^4\right)$$

If we write $w(x)$ as $a - bx^2 + cx^4$ then we can do a polynomial fit for the Dabbahu data. The parameter a tells us a value for w_0 — I got $\boxed{w_0 \approx 148 \text{ mm}}$. We can use b and c to calculate L as follows:

$$\frac{b}{c} = \frac{L^2}{2}$$

$$\Rightarrow \boxed{L \approx 45 \text{ km}}$$

Now we can use equation (3-127) to calculate the flexural rigidity D:

$$D = \frac{gL^4(\rho_c - \rho_{\text{mag}})}{256}$$

$$\Rightarrow \boxed{D \approx 6.5 \times 10^{19} \text{ Nm}}$$

We can use our value of D to solve for the plate thickness h:

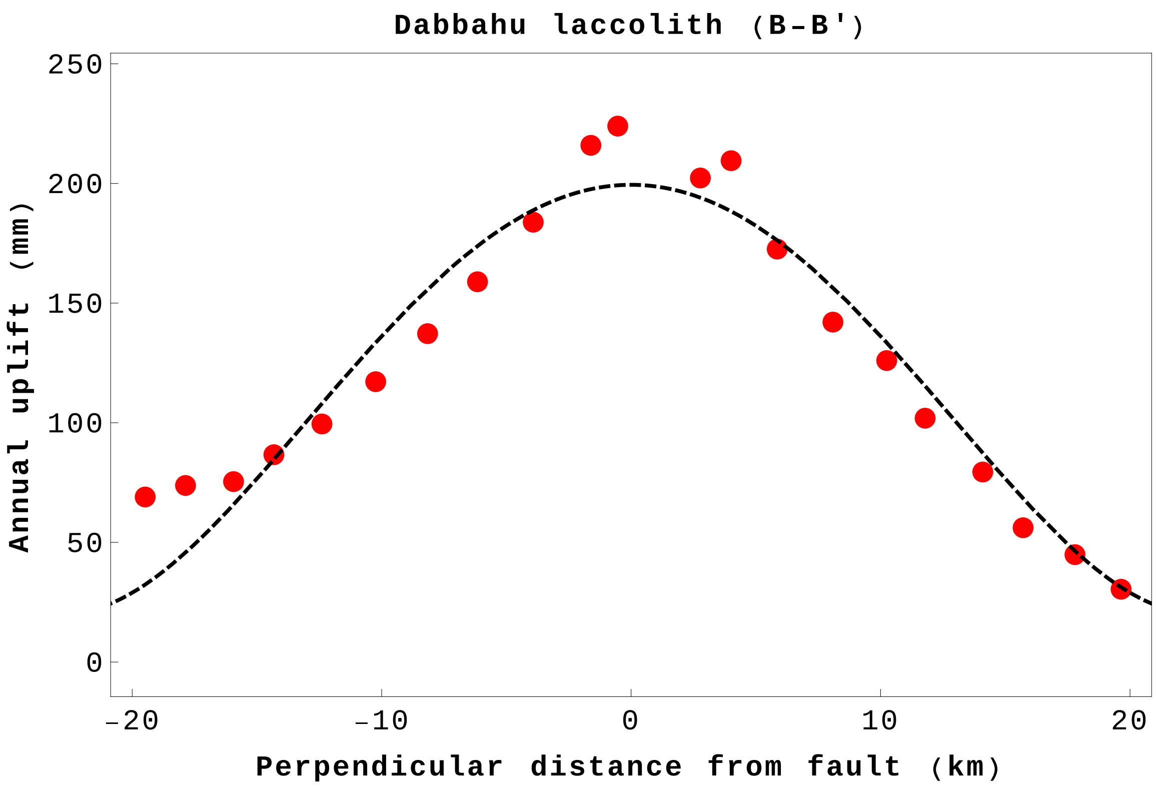
$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$\Rightarrow \boxed{h \approx 2.2 \text{ km}}$$

Finally we can solve for the magma pressure p:

$$w_0 = \frac{q}{D} = \frac{\rho_c gh - p}{D}$$

$$\Rightarrow \boxed{p \approx 57 \text{ MPa}}$$



Two kilometers or so to the roof of the laccolith seems reasonable given that Dabbahu sits on a plate triple junction (Somalian-Nubian-Arabian).

We expect dykes to nucleate at areas of great tensional stress. We know how the normal strain $\epsilon_{xx}(x, y)$ relates to the normal stress $\sigma_{xx}(x, y)$:

$$\sigma_{xx}(x, y) = \frac{E}{1 - \nu^2}\epsilon_{xx}(x, y)$$

And we know how $\epsilon_{xx}(x, y)$ relates to the deflection $w(x)$:

$$\epsilon_{xx}(x, y) = -y\frac{d^2w}{dx^2}$$

Now we can take the second derivative of the deflection $w(x)$ at evaluate that at $y = +h/2$ (the bottom of the plate):

$$\epsilon_{xx}(x) = \frac{qhL^2}{48D}\left(1 - \frac{12}{L^2}x^2\right)$$

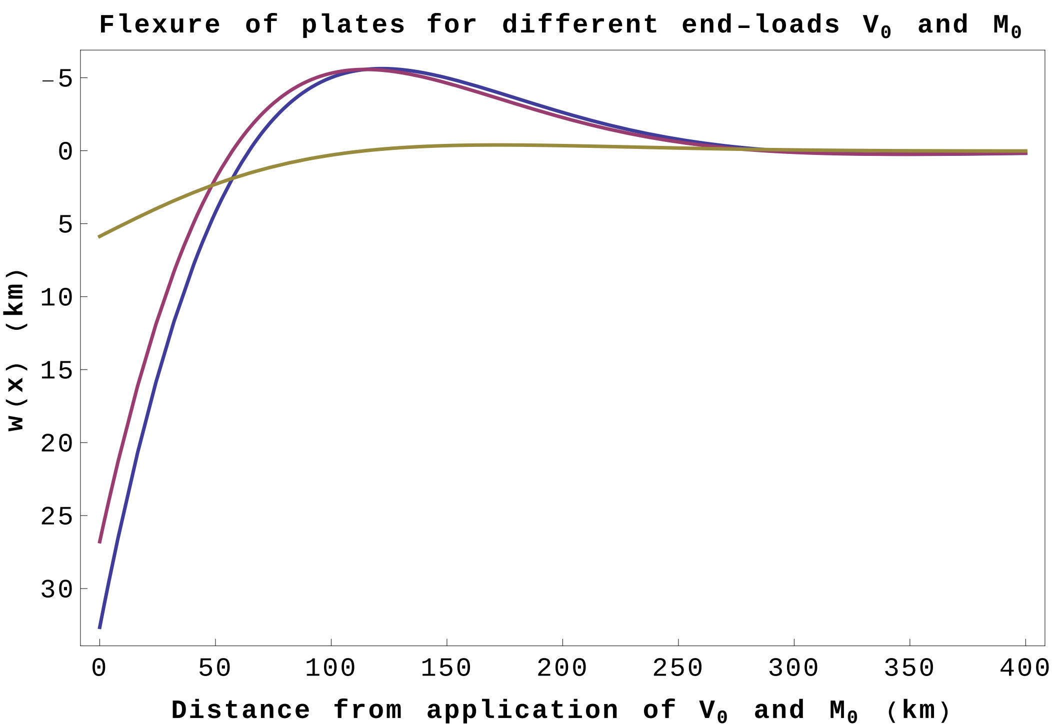
So the normal stresses $\sigma_{xx}(x)$ at the bottom of the plate can be found — recall that $D = Eh^3/12(1 - \nu^2)$:

$$\sigma_{xx}(x) = \frac{E}{1 - \nu^2}\epsilon_{xx}(x) = \frac{qL^2}{4h^2}\left(1 - \frac{12}{L^2}x^2\right)$$

We want the value of $x \in [-L/2, L/2]$ that produces the greatest *negative* (tensional) value of stress. We can see that $\sigma_{xx}(x)$ plots as a concave-downward parabola. Thus the greatest negative stresses must occur at the endpoints of the model domain — at $x = -L/2$ and $x = +L/2$.

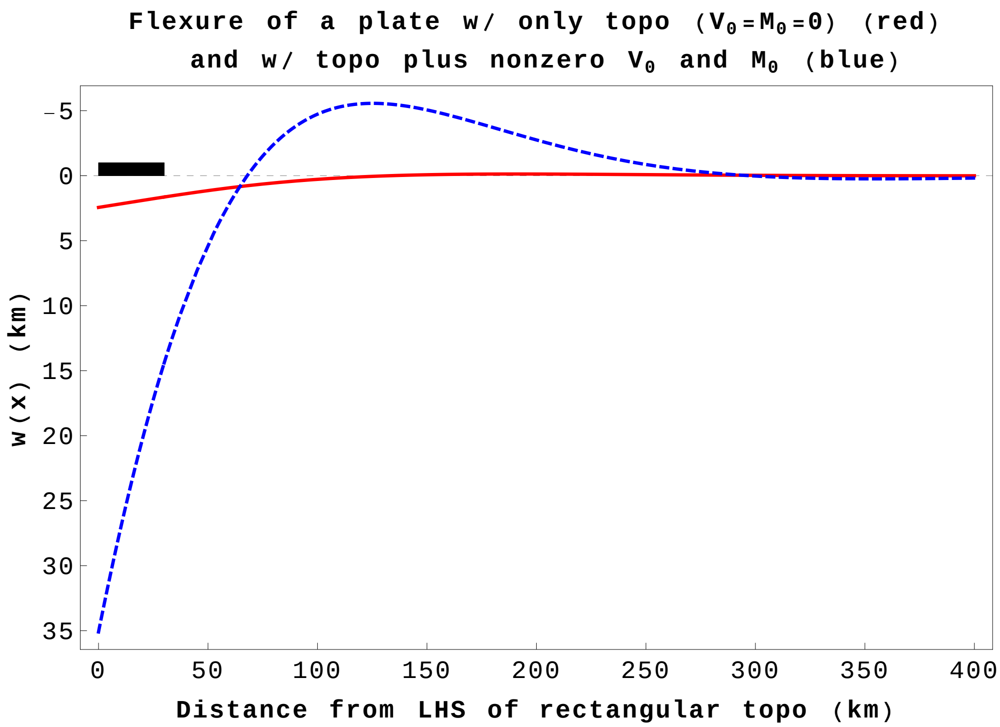
Problem 10 (Broken plate flexure)

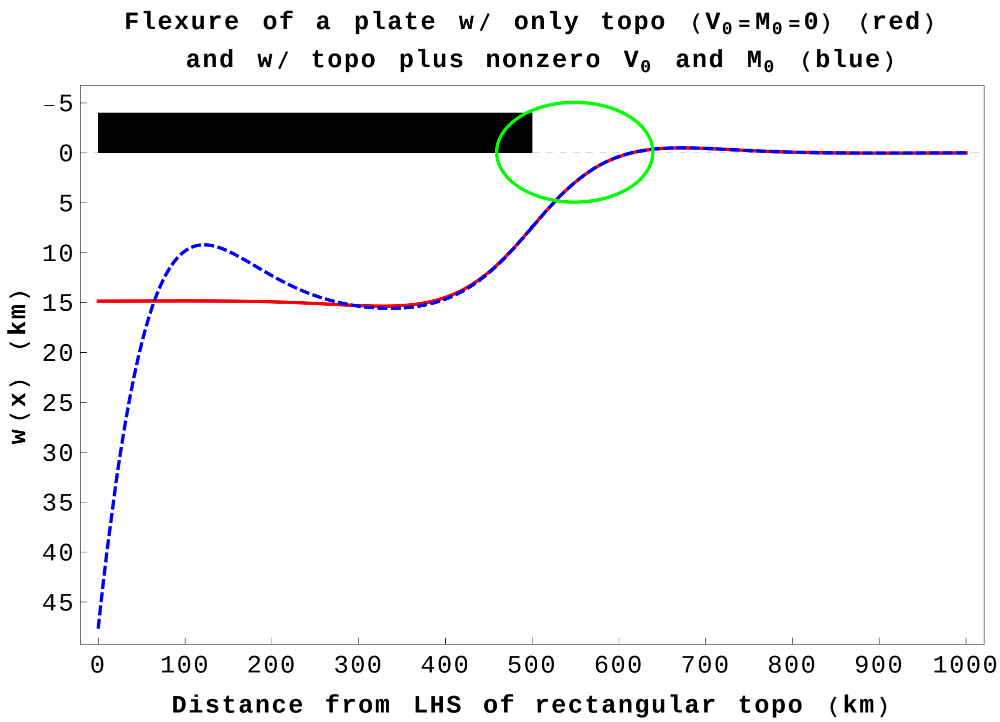
Blue: $V_0 = -1.5 \times 10^{12}$ N/m and $M_0 = -5.0 \times 10^{17}$ N. Purple: $V_0 = 0$ N/m and $M_0 = -5.0 \times 10^{17}$ N. Yellow: $V_0 = -1.5 \times 10^{12}$ N/m and $M_0 = 0$ N.



The flexural response depends upon the flexural rigidity D of the plate which scales as the *cube*(!) of the elastic plate thickness. In other words: A modest change to T_e greatly changes the flexural response of the plate.

Problem 11 (Adding topography)





We *cannot* use the shape of the Ganges Foredeep Basin (circled in green) to study slab-pull on the subducting Indian Plate. This is because the flexural response of the plate around the Foredeep Basin looks the same with or without loading at $x = 0$. Thus topography alone controls the shape of the Basin — at least for the case at hand.