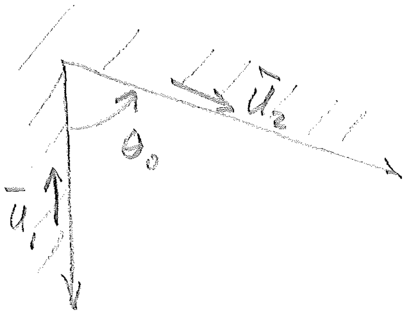
2/3/07, MR

Batchelor, §4.8. General (θ_0) corner flow problem



Assume $\psi = r f(\theta)$, $\nabla^2 \psi = 0$

$$f(\theta) = A \sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta$$

$$f'(\theta) = A\cos\theta - B\sin\theta + C(\sin\theta + \theta\cos\theta) + D(\cos\theta - \theta\sin\theta)$$

$$u_\theta = -f(\theta) = -\frac{\partial\psi}{\partial r}, \quad u_r = f'(\theta) = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

B.C. $\theta = 0$: $u_r = -U_1 = f'(0)$ (a) $\quad$ At $\theta = \theta_0$ $\quad u_r = U_2 = f'(\theta_0)$ (c)

$\qquad u_\theta = 0 = f(0)$ (b) $\qquad\qquad u_\theta = 0 = f(\theta_0)$ (d)

(b) $\Rightarrow B = 0$

(d) $\Rightarrow 0 = A\sin\theta_0 + C\theta_0\sin\theta_0 + D\theta_0\cos\theta_0$

(a) $\Rightarrow -U_1 = A + D$

(c) $\Rightarrow U_2 = A\cos\theta_0 + C(\sin\theta_0 + \theta_0\cos\theta_0) + D(\cos\theta_0 - \theta_0\sin\theta_0)$

use (a) to eliminate A:

(d) $\Rightarrow 0 = -(U_1 + D)\sin\theta_0 + C\theta_0\sin\theta_0 + D\theta_0\cos\theta_0$

(c) $\Rightarrow U_2 = -(U_1 + D)\cos\theta_0 + C(\theta_0\cos\theta_0 + \sin\theta_0) + D(\cos\theta_0 - \theta_0\sin\theta_0)$

$$\Rightarrow \quad U_1\sin\theta_0 = C\theta_0\sin\theta_0 + D(\theta_0\cos\theta_0 - \sin\theta_0)$$

$$U_2 + U_1\cos\theta_0 = C(\theta_0\cos\theta_0 + \sin\theta_0) - D\theta_0\sin\theta_0$$

$$\begin{pmatrix} U_1\sin\theta_0 \\ U_2 + U_1\cos\theta_0 \end{pmatrix} = \underbrace{\begin{pmatrix} \theta_0\sin\theta_0 & \theta_0\cos\theta_0 - \sin\theta_0 \\ \theta_0\cos\theta_0 + \sin\theta_0 & -\theta_0\sin\theta_0 \end{pmatrix}}_{\equiv M} \begin{pmatrix} C \\ D \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} -\theta_0\sin\theta_0 & \sin\theta_0 - \theta_0\cos\theta_0 \\ -\theta_0\cos\theta_0 - \sin\theta_0 & \theta_0\sin\theta_0 \end{pmatrix} \frac{1}{[\det M]}$$

$$\det M = -\theta_0^2\sin^2\theta_0 - (\theta_0^2\cos^2\theta_0 - \sin^2\theta_0)$$

$$= \sin^2\theta_0 - \theta_0^2$$

$$\Rightarrow \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} -\theta_0 \sin\theta_0 & \sin\theta_0 - \theta_0\cos\theta_0 \\ -\theta_0\cos\theta_0 - \sin\theta_0 & \theta_0 \sin\theta_0 \end{pmatrix} \begin{pmatrix} U_1 \sin\theta_0 \\ U_2 + U_1\cos\theta_0 \end{pmatrix} \frac{1}{\sin^2\theta_0 - \theta_0^2}$$

$$\Rightarrow C = \frac{-U_1\theta_0 \sin^2\theta_0 + (U_2 + U_1\cos\theta_0)(\sin\theta_0 - \theta_0\cos\theta_0)}{\sin^2\theta_0 - \theta_0^2}$$

$$\boxed{C = \frac{+U_1\theta_0 + U_2\theta_0\cos\theta_0 - (U_2 + U_1\cos\theta_0)\sin\theta_0}{\theta_0^2 - \sin^2\theta_0}}$$

$$D = \frac{-U_1\sin\theta_0(\theta_0\cos\theta_0 + \sin\theta_0) + (U_2 + U_1\cos\theta_0)\theta_0\sin\theta_0}{\sin^2\theta_0 - \theta_0^2}$$

$$= \frac{-U_1\sin^2\theta_0 + U_2\theta_0\sin\theta_0}{\sin^2\theta_0 - \theta_0^2} = \boxed{\frac{\sin\theta_0}{\theta_0^2 - \sin^2\theta_0}(U_1\sin\theta_0 - U_2\theta_0)}$$

$$A = -(U_1 + D) = -\left[\frac{U_1\theta_0^2 - \cancel{U_1\sin^2\theta_0} + \cancel{U_1\sin^2\theta_0} - U_2\theta_0\sin\theta_0}{\theta_0^2 - \sin^2\theta_0}\right]$$

$$\boxed{A = -\left[\frac{U_1\theta_0^2 - U_2\theta_0\sin\theta_0}{\theta_0^2 - \sin^2\theta_0}\right]}$$

Note: if $U_2 = 0$, then

$$A = \frac{U_1}{\theta_0^2 - \sin^2\theta_0}(-\theta_0^2)$$

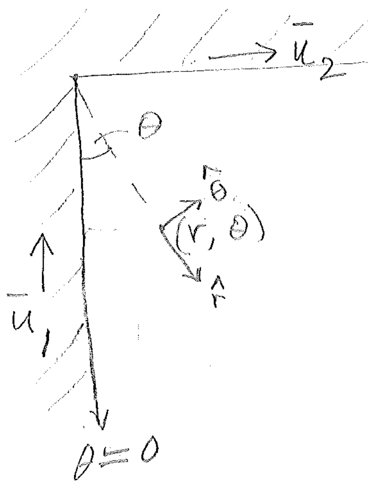
$$C = \frac{U_1}{\theta_0^2 - \sin^2\theta_0}(\theta_0 - \cos\theta_0\sin\theta_0)$$

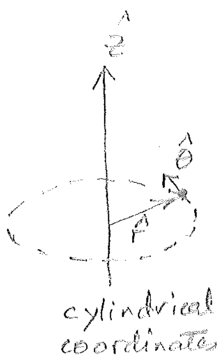
$$D = \frac{U_1}{\theta_0^2 - \sin^2\theta_0}\sin^2\theta_0$$

which is the same as eqn 4.8.26 in Batchelor

(90^0) (Batchelor, §4.8) M.R. 2/2/07

Corner flow, general boundary conditions:





Assume:

$\psi = r f(\theta)$, $\nabla^2\psi = 0$

$\Rightarrow f(\theta) = A\sin\theta + B\cos\theta + C\theta\sin\theta + D\theta\cos\theta$

& $f'(\theta) = A\cos\theta - B\sin\theta + C(\sin\theta + \theta\cos\theta) + D(\cos\theta - \theta\sin\theta)$

$\bar{u} = (u_r, u_\theta)$

$u_\theta = -\frac{\partial\psi}{\partial r}$ $\quad u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$

$u_\theta = -f(\theta)$ $\quad u_r = f'(\theta)$

Choose boundary conditions:

At $\theta = 0$: (a) $u_r = -u_1 \Rightarrow f'(0) = -u_1$

(b) $u_\theta = 0 \Rightarrow f(0) = 0$

At $\theta = \frac{\pi}{2}$: $u_r = +u_2 = f'(\pi/2)$ (c)

$u_\theta = 0 = f(\pi/2)$ (d)

(b) $\Rightarrow B = 0$

(d) $\Rightarrow 0 = A + C\frac{\pi}{2}$

(a) $\Rightarrow -u_1 = A + D$

(c) $\Rightarrow +u_2 = C - D\frac{\pi}{2}$

eliminate A, C: $A = -(u_1 + D)$ ((a))

$C = u_2 + \frac{D\pi}{2}$ ((c))

$$\Rightarrow f(\theta) = -(u_1 + D)\sin\theta + \left(u_2 + \frac{D\pi}{2}\right)(\sin\theta)\,\theta + D\theta\cos\theta$$

$$= (u_2\theta - u_1)\sin\theta + D\left(-\sin\theta + \theta\frac{\pi}{2}\sin\theta + \theta\cos\theta\right)$$

Note: we still need to use (d): $f(\pi/2) = 0$

$$\Rightarrow 0 = -(u_1 + D) + \left(u_2 + \frac{D\pi}{2}\right)\frac{\pi}{2} = -u_1 + u_2\frac{\pi}{2} + D\left(\frac{\pi^2}{4} - 1\right)$$

$$\Rightarrow D = \left(u_1 - u_2\frac{\pi}{2}\right)/\left(\frac{\pi^2}{4} - 1\right) = 4(u_1 - u_2\pi/2)/(\pi^2 - 4)$$

$$\Rightarrow f(\theta) = -\left(u_1 + \frac{u_1}{\frac{\pi^2}{4} - 1} - \frac{u_2\,\pi/2}{\pi^2/4 - 1}\right)\sin\theta + \left(u_2 + \frac{u_1\pi/2}{\pi^2/4 - 1} - \frac{u_2\frac{\pi^2}{4}}{\pi^2/4 - 1}\right)\theta\sin\theta$$

$$+ \left(\frac{u_1 - u_2\pi/2}{\frac{\pi^2}{4} - 1}\right)\theta\cos\theta$$

Simplify coefficients:

$\sin\theta$ term: $$-\frac{\left(u_1\left(\frac{\pi^2}{4}\right) - u_2\pi/2\right)}{\left(\pi^2/4 - 1\right)} = -\frac{\pi}{2}\frac{\left(u_1\pi/2 - u_2\right)}{\left(\pi^2/4 - 1\right)}$$

$\theta\sin\theta$ term: $$\left(\frac{u_1\pi/2 - u_2}{\pi^2/4 - 1}\right)$$

$\Rightarrow$ Collecting terms:

$$\boxed{f(\theta) = \left(\frac{u_1\pi/2 - u_2}{\pi^2/4 - 1}\right)(\theta - \pi/2)\sin\theta + \left(\frac{u_1 - u_2\pi/2}{\pi^2/4 - 1}\right)\theta\cos\theta}$$

<u>Note</u>: If $u_2 = 0$,

$$f(\theta) = \frac{u_1}{\left(\pi^2/4 - 1\right)}\left[\frac{\pi}{2}\theta\sin\theta - \frac{\pi^2}{4}\sin\theta + \theta\cos\theta\right]$$ $\rightarrow$ consistent with eqn. 4.8.27 in Batchelor, p. 226

<u>If</u>: $u_2 = u_1\pi/2$, $f(\theta) = \cancel{-\theta\cos\theta}$, which must be the boundary conditions in the Kaminski-Ribe EPSL paper

In this case,

$$u_\theta = -f(\theta) = \theta\cos\theta$$
$$u_r = f'(\theta) = -(\cos\theta - \theta\sin\theta) = \theta\sin\theta - \cos\theta$$

$$f(\theta) = \frac{\left(u_1 - u_1\pi^2/2^2\right)}{\left(\frac{\pi^2}{4} - 1\right)}\theta\cos\theta = \underline{-u_1}(\theta\cos\theta)$$

K & R 6^3 '02 $= u_1$

$$f(\theta) = -\frac{2u}{\pi}(\theta\cos\theta)$$

$$u_\theta = -f(\theta) \Rightarrow u_\theta + f(\theta) = 0$$