

Math Review

- Gradient:

Consider a function that maps a coordinate (e.g. from \mathbb{R}^3) to a scalar value:

$$F: \mathbb{R}^3 \rightarrow \mathbb{R}, (x, y, z) \mapsto F(x, y, z) \quad \text{sometimes called a "scalar field"}$$

F could be an elevation above a surface, or the density of a material at a specific point. The gradient is an operator which turns such a function into a vector function describing the rate of change of F . It is written as $\text{grad } F$ or $\vec{\nabla} F$, called the gradient of F :

$$\text{grad } F = \vec{\nabla} F(x, y, z) = \left(\frac{\partial F(x, y, z)}{\partial x}, \frac{\partial F(x, y, z)}{\partial y}, \frac{\partial F(x, y, z)}{\partial z} \right) \in \mathbb{R}^3.$$

If F is a constant function, $\vec{\nabla} F = \vec{0}$ everywhere.

- Divergence:

Imagine drawing a vector at each point in a coordinate system, which can be described by a function that maps coordinates to vector values:

$$\vec{V}: \mathbb{R}^3 \rightarrow \mathbb{R}^3, (x, y, z) \mapsto \vec{V}(x, y, z) = (V_x(x, y, z), V_y(x, y, z), V_z(x, y, z))^T \in \mathbb{R}^3.$$

This is sometimes called a "vector field". The divergence of \vec{V} , written as $\text{div } \vec{V}$, or $\vec{\nabla} \cdot \vec{V}$, is the sum of the rates of change of the componentwise functions along their dimensions:

$$\text{div } \vec{V} = \vec{\nabla} \cdot \vec{V}(x, y, z) = \frac{\partial V_x(x, y, z)}{\partial x} + \frac{\partial V_y(x, y, z)}{\partial y} + \frac{\partial V_z(x, y, z)}{\partial z} \in \mathbb{R}.$$

Note that the divergence is a scalar function. Physically, it can be considered a rate of transfer per unit of volume (see Fowler, p. 617).

- Laplacian:

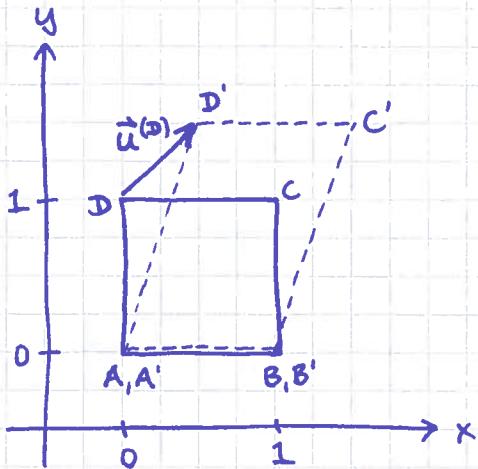
When we calculate the divergence of a gradient, we get the Laplacian operator $\vec{\nabla}^2$:

$$(\vec{\nabla}^2 F)(x, y, z) := \vec{\nabla} \cdot (\vec{\nabla} F)(x, y, z) = \frac{\partial^2 F(x, y, z)}{\partial x^2} + \frac{\partial^2 F(x, y, z)}{\partial y^2} + \frac{\partial^2 F(x, y, z)}{\partial z^2} \in \mathbb{R}.$$

The Laplacian for a vector-valued function is applied componentwise:

$$(\vec{\nabla}^2 \vec{V})(x, y, z) := ((\nabla^2 V_x)(x, y, z), (\nabla^2 V_y)(x, y, z), (\nabla^2 V_z)(x, y, z)).$$

Strain



Imagine the square ABCD being deformed into the shape A'B'C'D'. At each point of the original shape, we can calculate a translation vector $\vec{u}^{(P)}$ which takes P to P'. In our case, $\vec{u}^{(A)} = \vec{u}(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\vec{u}^{(B)} = \vec{u}(1,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\vec{u}^{(C)} = \vec{u}(1,1) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, and $\vec{u}^{(D)} = \vec{u}(0,1) = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$. In general, at any point in this transformation, $\vec{u}(x,y) = \begin{pmatrix} y/2 \\ y/2 \end{pmatrix}$.

As usual, we can decompose $\vec{u}(x,y)$ into componentwise functions:

$$\vec{u}(x,y) = (u_x(x,y), u_y(x,y)) \text{ where } u_x(x,y) = u_y(x,y) = y/2 \text{ in our example.}$$

The Jacobian matrix J for \vec{u} is a 2×2 matrix with components $J_{ij} = \partial u_i / \partial j$, so:

$$J = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}.$$

The strain tensor is a 2×2 matrix ε with components $\varepsilon_{ij} = \frac{1}{2}(J_{ij} + J_{ji} + J_{ii}J_{jj})$.

For $J_{ii} \ll 1$ the ε_{ij} can be approximated by $\varepsilon_{ij} = \frac{1}{2}(J_{ij} + J_{ji})$ (infinitesimal strain).

This approximation ensures that ε will be a symmetric tensor no matter what J looks like. In our case,

$$\varepsilon = \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

The diagonal elements ε_{ii} are called "normal strains", and the other elements are called "shear strains".

Decomposition of Displacement - Gradient Tensor

Recall that a matrix A is called symmetrical when $A^T = A$ and antisymmetrical when $A^T = -A$. In general, any square matrix A can be split into symmetrical and antisymmetrical components using the identity

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symmetric}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{antisymmetric}}$$

(continued)

Strain (continued)

The Jacobian matrix in our example can be written as a gradient using

$\vec{\nabla} \vec{u} = \begin{pmatrix} \vec{\nabla} u_x \\ \vec{\nabla} u_y \end{pmatrix} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{pmatrix} = J$. In this form, $\vec{\nabla} \vec{u}$ is called a 2nd rank displacement-gradient tensor. E then becomes the symmetric part of $\vec{\nabla} \vec{u}$ when it is decomposed into symmetric (deformation) and antisymmetric (rotation) components. In three dimensions, this decomposition looks like this:

$$\vec{u}(x, y, z) = \underbrace{\begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix}}_{\text{strain}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & \theta_z & -\theta_y \\ -\theta_z & 0 & \theta_x \\ \theta_y & -\theta_x & 0 \end{pmatrix}}_{\text{rotation}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

In three-dimensional solids, strain also causes a change in volume:

$$\text{Original volume } V = dx dy dz$$

$$\text{"Strained" volume } V + dV = (1 + \epsilon_{xx})dx (1 + \epsilon_{yy})dy (1 + \epsilon_{zz})dz$$

$$\begin{aligned} \text{Cubical dilatation } \Delta &= \frac{\text{change in } V}{\text{original } V} = \frac{V + dV - V}{V} = \frac{(1 + \epsilon_{xx})dx (1 + \epsilon_{yy})dy (1 + \epsilon_{zz})dz - dx dy dz}{dx dy dz} \\ &= \frac{dx dy dz + (\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} + \epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}\epsilon_{zz} + \epsilon_{zz}\epsilon_{xx} + \epsilon_{xx}\epsilon_{yy}\epsilon_{zz})dx dy dz - dx dy dz}{dx dy dz} \\ &= \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} + \epsilon_{xx}\epsilon_{yy} + \epsilon_{yy}\epsilon_{zz} + \epsilon_{zz}\epsilon_{xx} + \epsilon_{xx}\epsilon_{yy}\epsilon_{zz}. \end{aligned}$$

Under infinitesimal strain, where products of strains can be neglected,

$$\text{we get } \boxed{\Delta = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = \text{tr } E}.$$

Furthermore, since $\epsilon_{xx} = \frac{\partial u_x}{\partial x}$, $\epsilon_{yy} = \frac{\partial u_y}{\partial y}$, $\epsilon_{zz} = \frac{\partial u_z}{\partial z}$, we can also write $\boxed{\Delta = \vec{\nabla} \cdot \vec{u}}$.

(divergence of \vec{u})

The Relationship between Stress and Strain

Stresses are difficult to measure, but strains are more obvious. It would be nice if we had a formula to calculate stress when we know the strain. This formula is known as Hooke's Law, and in its most general form written as $\sigma_{ij} = C_{ijkl} \epsilon_{kl}$ (using Einstein's summation convention — this is really to be read as

$$\sigma_{ij} = \sum_{k=1}^3 \sum_{l=1}^3 C_{ijkl} \epsilon_{kl}.$$

Each stress component can depend on a linear

combination of all strains. Given that due to symmetry we have six unique strains and six unique stresses, C_{ijkl} would be a 4th-rank tensor containing up to 36 unique coefficients. Clearly not desirable.

For linear (isotropic), elastic solids, we only need two coefficients (moduli), λ and μ , called Lamé parameters. Hooke's Law then becomes

$$\sigma_{xx} = (\lambda + 2\mu) \epsilon_{xx} + \lambda \epsilon_{yy} + \lambda \epsilon_{zz} = \lambda \Delta + 2\mu \epsilon_{xx},$$

$$\sigma_{yy} = \lambda \epsilon_{xx} + (\lambda + 2\mu) \epsilon_{yy} + \lambda \epsilon_{zz} = \lambda \Delta + 2\mu \epsilon_{yy},$$

$$\sigma_{zz} = \lambda \epsilon_{xx} + \lambda \epsilon_{yy} + (\lambda + 2\mu) \epsilon_{zz} = \lambda \Delta + 2\mu \epsilon_{zz},$$

$$\sigma_{xy} = \sigma_{yx} = 2\mu \epsilon_{xy}, \quad \sigma_{xz} = \sigma_{zx} = 2\mu \epsilon_{xz}, \quad \sigma_{yz} = \sigma_{zy} = 2\mu \epsilon_{yz},$$

$$\text{or } \boxed{\sigma_{ij} = \lambda \Delta \delta_{ij} + 2\mu \epsilon_{ij}}.$$

Other ways of expressing this relationship are Young's modulus, Poisson's ratio, and the bulk modulus. Young's modulus and Poisson's ratio describe strains caused by longitudinal stresses and are derived from λ and μ by

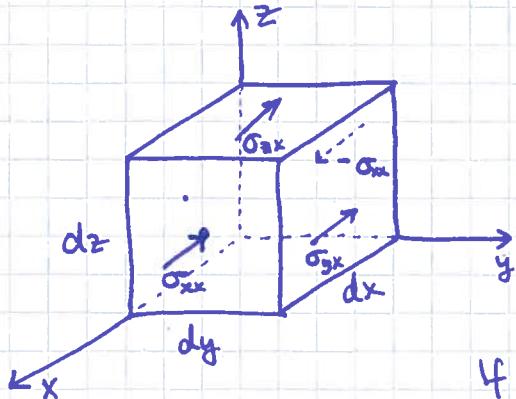
$$E = \underbrace{\frac{\sigma_{xx}}{\epsilon_{xx}}}_{\text{Young's modulus}} = \frac{(3\lambda + 2\mu)\mu}{\lambda + \mu} \quad \text{and} \quad \sigma = \underbrace{\frac{-\epsilon_{yy}}{\epsilon_{xx}} = \frac{-\epsilon_{zz}}{\epsilon_{xx}} = \frac{\lambda}{2(\lambda + \mu)}}_{\text{Poisson's ratio}}.$$

($\sim 60\text{-}70 \text{ GPa}$ for rocks)

The bulk modulus or incompressibility under hydrostatic pressure is given by

$$K = \frac{\text{pressure}}{\text{compression}} = \frac{\text{pressure}}{-\text{dilatation}} = \frac{P}{-\Delta} = \lambda + \frac{2}{3}\mu.$$

Equations of Motion



going back to our stressed cube, we notice that the stress tensor varies with position. Differences in forces between the front and back faces, for example, can add to a nonzero net force in the x -direction, creating motion in accordance with Newton's 2nd Law. If we treat each of the components of the stress tensor as a function in x , y , and z , we can take partial derivatives. The difference in σ_{xx} between the front and back faces is $\frac{\partial \sigma_{xx}}{\partial x} dx$, and the diff. between the top and bottom faces is $\frac{\partial \sigma_{zz}}{\partial z} dz$. The net forces (stress times area) are $(\frac{\partial \sigma_{xx}}{\partial x} dx) dy dz$ and $(\frac{\partial \sigma_{zz}}{\partial z} dz) dx dy$, respectively. Together with the forces on the side faces, the total net force in the x -direction is $F_x = (\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}) dx dy dz$, where $dx dy dz$ is the volume under consideration. Writing mass as density times volume, we get:

$$F_x = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) dx dy dz = m \cdot a_x = \rho dx dy dz \cdot \frac{\partial^2 u_x}{\partial t^2}.$$

Substituting our strains from Hooke's Law for the stresses, and simplifying, we get

$$\begin{aligned} \rho \frac{\partial^2 u_x}{\partial t^2} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \frac{\partial}{\partial x} (\lambda \Delta + 2\mu \epsilon_{xx}) + \frac{\partial}{\partial y} (2\mu \epsilon_{yy}) + \frac{\partial}{\partial z} (2\mu \epsilon_{zz}) \\ &= \frac{\partial}{\partial x} \left(\lambda \Delta + 2\mu \frac{\partial u_x}{\partial x} \right) + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[\mu \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) \right] \\ &= \lambda \frac{\partial \Delta}{\partial x} + 2\mu \frac{\partial^2 u_x}{\partial x^2} + \mu \frac{\partial^2 u_x}{\partial x \partial y} + \mu \frac{\partial^2 u_x}{\partial y \partial z} + \mu \frac{\partial^2 u_z}{\partial x \partial z} + \mu \frac{\partial^2 u_k}{\partial z^2} \\ &= \lambda \underbrace{\frac{\partial \Delta}{\partial x}}_{\Delta} + \mu \underbrace{\frac{\partial}{\partial x} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right)}_{\nabla^2 u_x} + \mu \underbrace{\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right)}_{\nabla^2 u_x} \\ &= \lambda \frac{\partial \Delta}{\partial x} + \mu \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u_x \end{aligned}$$

or

$$\rho \frac{\partial^2 u_x}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u_x$$

for the x -direction

Equations of Motion (continued)

Similarly, we can treat the y - and z -direction to obtain

$$\rho \frac{\partial^2 u_x}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u_x$$

$$\rho \frac{\partial^2 u_y}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial y} + \mu \nabla^2 u_y$$

$$\rho \frac{\partial^2 u_z}{\partial t^2} = (\lambda + \mu) \frac{\partial \Delta}{\partial z} + \mu \nabla^2 u_z$$

$$\rho \frac{\partial^2}{\partial t^2} (u_x + u_y + u_z) = (\lambda + \mu) \left(\frac{\partial \Delta}{\partial x} + \frac{\partial \Delta}{\partial y} + \frac{\partial \Delta}{\partial z} \right) + \mu \nabla^2 (u_x + u_y + u_z)$$

Differentiating $\partial u_x / \partial x$, $\partial u_y / \partial y$, $\partial u_z / \partial z$:

$$\rho \frac{\partial^2}{\partial t^2} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) = (\lambda + \mu) \underbrace{\left(\frac{\partial^2 \Delta}{\partial x^2} + \frac{\partial^2 \Delta}{\partial y^2} + \frac{\partial^2 \Delta}{\partial z^2} \right)}_{\nabla^2 \Delta} + \mu \nabla^2 \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right)$$

$$\Rightarrow \rho \frac{\partial^2 \Delta}{\partial t^2} = (\lambda + \mu) \nabla^2 \Delta + \mu \nabla^2 \Delta \quad \text{or} \quad \boxed{\frac{\partial^2 \Delta}{\partial t^2} = \left(\frac{\lambda + 2\mu}{\rho} \right) \nabla^2 \Delta} \quad \text{wave equation}$$

This is a wave eqn. of a Δ (dilatational) disturbance transmitted at a speed of

$$\alpha = \sqrt{v^2} = \sqrt{\frac{\lambda + 2\mu}{\rho}}.$$

Check the units of α : λ and μ are both pressures ($N \cdot m^{-2}$ or $kg \cdot m^{-1} \cdot s^{-2}$)

ρ is density ($kg \cdot m^{-3}$)

$$\text{Then } \sqrt{\frac{kg \cdot m^{-1} \cdot s^{-2}}{kg \cdot m^{-3}}} = \sqrt{m^2 \cdot s^{-2}} = m \cdot s^{-1} \text{ or } \underline{m/s} \text{ which is velocity. } \square$$

This purely dilatational wave is called the primary, longitudinal, or P-wave.

The other wave (see Fowler, p. 627) is a "disturbance" in the curl of \vec{u} , which equals twice the rotation ($\theta_x, \theta_y, \theta_z$). This wave, which travels at a speed of $\beta = \sqrt{\mu/\rho}$, is called the secondary, shear, or S-wave.

Note that because $\lambda > 0$ and $\mu > 0$ it follows that $\alpha > \beta$, so the P-wave always travels faster than the S-wave in the same medium!