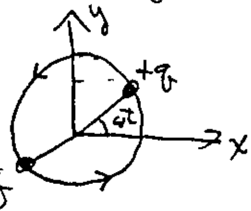
11.1. Radiation by a rotating electric dipole



$$\vec{p}(t) = p_0 \cos(\omega t)\,\hat{x} + p_0 \sin(\omega t)\,\hat{y}$$

Using the principle of superposition, we can add the dipole radiation associated with the dipole along x to that for the dipole along y.

For a monochromatic oscillating dipole, we have the complex amplitude for the electric field

$$\underset{\sim}{\vec{E}} = +\frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} \underset{\sim}{\vec{p}}_{0\perp} \quad \text{where} \quad \vec{p}(t) = \mathrm{Re}\left(\underset{\sim}{\vec{p}_0}\, e^{-i\omega t}\right)$$

$\Leftarrow$ component $\perp$ to direction of observation

$$= +\frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} \left(\underset{\sim}{\vec{p}_0} - \hat{r}\,(\hat{r}\cdot\underset{\sim}{\vec{p}_0})\right)$$

In this case: $\vec{p}_0(t) = \mathrm{Re}\left(p_0 e^{-i\omega t}\,\hat{x} + \underset{\uparrow}{i} p_0 e^{-i\omega t}\,\hat{y}\right)$

90° out of phase

$$\Rightarrow \underset{\sim}{\vec{E}} = +\frac{1}{4\pi\epsilon_0} k^2 \frac{e^{ikr}}{r} p_0 \left\{ \left(\hat{x} - \hat{r}(\hat{r}\cdot\hat{x})\right) + i\left(\hat{y} - \hat{y}(\hat{r}\cdot\hat{y})\right)\right)$$

From the inside back cover of Griffiths, we can express the Cartesian basis vectors in terms of those in spherical coordinates (which depends upon the point of observation)

$$\hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$$

$$\Rightarrow \hat{x} - \hat{r}(\hat{x}\cdot\hat{r}) = \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$$

$$\hat{y} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$$

$$\Rightarrow \hat{y} - \hat{r}(\hat{y}\cdot\hat{r}) = \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}$$

Plug into the expression for the electric field complex amplitude

$$\Rightarrow \tilde{\vec{E}} = \frac{1}{4\pi\epsilon_0} k^2 p_0 \frac{e^{ikr}}{r}\Big((\cos\theta\cos\phi + i\cos\theta\sin\phi)\,\hat{\theta} + (-\sin\phi + i\cos\phi)\,\hat{\phi}\Big)$$

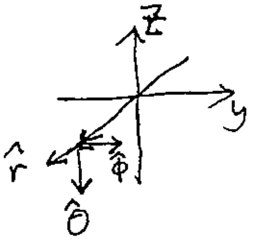
$$= \frac{1}{4\pi\epsilon_0} k^2 p_0 \frac{e^{ikr}}{r}\Big(\cos\theta\,\underbrace{(\cos\phi + i\sin\phi)}_{=e^{i\phi}}\,\hat{\theta} + i\,\underbrace{(\cos\phi + i\sin\phi)}_{=e^{i\phi}}\,\hat{\phi}\Big)$$

$$\Rightarrow \boxed{\tilde{\vec{E}} = \frac{1}{4\pi\epsilon_0} k^2 p_0 \left(\cos\theta\,\hat{\theta} + i\,\hat{\phi}\right)\frac{e^{i(kr+\phi)}}{r}}$$

The phase of the wave is $kr + \phi = \omega\frac{r}{c} + \phi$, where (r, ϕ) are the coordinates of the observer (in the radiation zone). This makes sense: The first term is the retarded time effect. The phase of oscillation should depend on the phi coordinate because the dipole **is** rotating in the ϕ-direction. Thus observers at different ϕ will see the wave at a different phase of the oscillation.

(b)

<u>On the x-axis</u>: $\theta = \frac{\pi}{2}$, $\phi = 0$, $y = z = 0 \Rightarrow r = x$



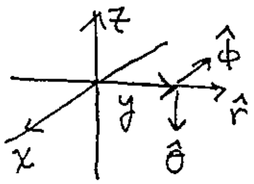
$$\hat{r} = \hat{x}, \quad \hat{\theta} = -\hat{z}, \quad \hat{\phi} = \hat{y}$$

$$\Rightarrow \vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \, \mathrm{Re}\left(i\,\hat{y}\,\frac{e^{i(kx-\omega t)}}{x}\right)$$

$$\Rightarrow \boxed{\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \frac{\sin(\omega t - kx)}{x}\,\hat{y}}$$

Linear polarization along $\hat{y}$

<u>On the y-axis</u>: $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$, $x = z = 0 \Rightarrow r = y$



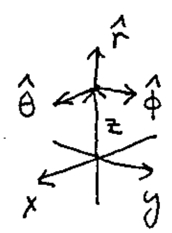
$$\hat{r} = \hat{y}, \quad \hat{\theta} = -\hat{z}, \quad \hat{\phi} = -\hat{x}$$

$$\Rightarrow \vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \, \mathrm{Re}\left(-i\,\hat{x}\,\frac{e^{i(ky-\omega t)}}{y}\,\underbrace{e^{i\pi/2}}_{=i}\right)$$

$$\Rightarrow \boxed{\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \frac{\cos(\omega t - ky)}{y}\,\hat{x}}$$

Linear polarization along $\hat{x}$

<u>On the z-axis</u> $\theta = 0$, $\phi = 0$, $x = y = 0 \Rightarrow r = z$



$$\Rightarrow \hat{r} = \hat{z}, \quad \hat{\theta} = \hat{x}, \quad \hat{\phi} = \hat{y}$$

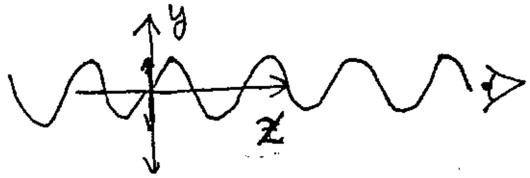
$$\Rightarrow \vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \, \mathrm{Re}\left\{(\hat{x} + i\hat{y})\frac{e^{ikz}}{z} e^{-i\omega t}\right\}$$

$$\Rightarrow \boxed{\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \left(\frac{\cos(\omega t - kz)\,\hat{x} + \sin(\omega t - kz)\,\hat{y}}{z}\right)}$$

Right ~~hand~~ circular polarization

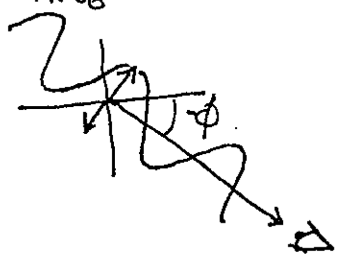
Comment on wave polarization

An observer on the x-axis sees only the projection of the rotating dipole along the y-axis, and vice-versa for observers on the y-axis



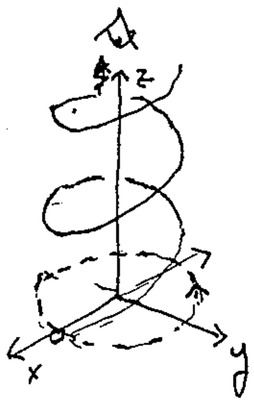
Generally, anywhere in the x-y plane

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} k^2 p_0 \cos\left(\omega t - k\sqrt{x^2+y^2} + \phi\right) \hat{\phi}$$



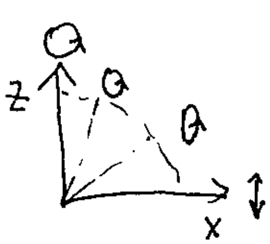
Linear polarization along $\hat{\phi}$, phase of oscillation depends on ϕ

On the z-axis



Traveling helix (right hand cork screw)
$\Rightarrow$ R.H. circular polarization

For an arbitrary point not along one of the axes the polarization is <u>elliptical</u> since the observer does not see an equal projection of oscillations along the two directions $\perp$ to the direction of observation



(c) The intensity

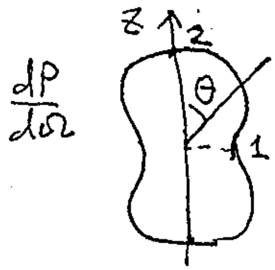
$$\langle S \rangle = c\frac{\epsilon_0}{2}\,\underset{\sim}{\vec{E}}^{*}\cdot\underset{\sim}{\vec{E}} = c\frac{\epsilon_0}{2}\left(\frac{1}{16\pi^2\epsilon_0^2}\,k^4 p_0^2\,(\cos^2\theta+1)\,\frac{1}{r^2}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{ck^4p_0^2}{8\pi}\right)\frac{1}{r^2}\,(1+\cos^2\theta)$$

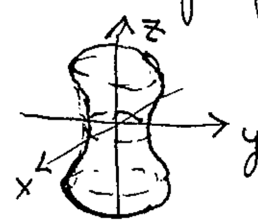
The power radiated into the element of area subtended by the solid angle $d\Omega$: $dP = \langle S\rangle dA = \langle S\rangle r^2 d\Omega$

$\Rightarrow$ differential $\frac{\text{Power}}{\text{solid angle}}$ $\boxed{\frac{dP}{d\Omega} = \langle S\rangle r^2 = \frac{1}{4\pi\epsilon_0}\left(\frac{ck^4p_0^2}{8\pi}\right)(1+\cos^2\theta)}$

Polar plot: At $\theta = 0$ $\frac{dP}{d\Omega}$ twice that at $\theta = \frac{\pi}{2}$. $\cos^2\theta$ increases as θ changes from $\frac{\pi}{2}$



Azimuthally symmetric



(d) Total radiated power into all 4π steradians:

$$P_{total} = \int \frac{dP}{d\Omega}\,d\Omega = \int \frac{dP}{d\Omega}\,2\pi\sin\theta\,d\theta = \frac{1}{4\pi\epsilon_0}\left(\frac{ck^4p_0^2}{8\pi}\right)2\pi\int_0^{\pi}(1+\cos^2\theta)\times\sin\theta\,d\theta$$

Let $\mu = \cos\theta \Rightarrow d\mu = -\sin\theta\,d\theta$

$$\therefore \int_0^{\pi}(1+\cos^2\theta)\sin\theta\,d\theta = \int_{-1}^{1}(1+\mu^2)\,d\mu = \left(\mu+\frac{\mu^3}{3}\right)\Big|_{-1}^{1} = \frac{8}{3}$$

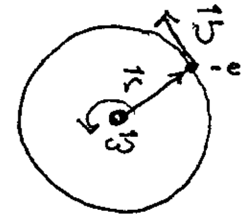
$$\therefore \boxed{P_{total} = \frac{1}{4\pi\epsilon_0}\left(\frac{2}{3}ck^4p_0^2\right)}$$

Factor of two because we have the superposition of two dipoles

(No interference term)
: orthogonal fields

11.2. Decay of a classical atom

Hydrogen: Electron bound to a proton
(Circular orbit)



Instantaneous energy: E = Kinetic + potential

$$\Rightarrow E = \frac{1}{2} m v^2 - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r}$$

Instantaneous force: $|\vec{F}_{centripetal}| = |\vec{F}_{coulomb}|$

$$\Rightarrow \frac{mv^2}{r} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2}$$

$$\therefore E = \frac{-1}{4\pi\epsilon_0} \frac{e^2}{2r}$$

(Also follows from Virial Theorem: $\langle \text{Kinetic} \rangle = -\frac{1}{2} \langle rF \rangle$)

As the electron orbits the nucleus it will radiate electromagnetic energy since it is constantly accellerating. From problem 1, $P_{rad} = \frac{1}{4\pi\epsilon_0} \frac{2}{3} c k^4 p_0^2$, $p_0 = er$

$$\Rightarrow \frac{dE}{dt} = -\frac{d}{dt}\left(\frac{1}{4\pi\epsilon_0}\frac{e^2}{2r}\right) = +\frac{1}{4\pi\epsilon_0}\frac{e^2}{2r^2}\frac{dr}{dt} = \underset{\substack{\uparrow \\ \text{decrease} \\ \text{in energy}}}{-} P_{rad}$$

$$= -P_{rad} = \frac{-1}{4\pi\epsilon_0} \frac{2}{3} \frac{\omega^4}{c^3} e^2 r^2$$

$$\Rightarrow \frac{dr}{dt} = -\frac{4}{3}\frac{\omega^4}{c^3} r^4$$

The instantaneous angular frequency $\omega = \frac{v}{r} \Rightarrow \omega = \sqrt{\frac{e^2}{4\pi\epsilon_0 m r^3}}$

$$\therefore \frac{dr}{dt} = -\frac{4}{3} c \left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2 \frac{1}{r^2}$$

(Next Page)

Define the "classical energy radius"

$$r_{class} = \frac{e^2}{4\pi\epsilon_0 mc^2} = 2.8 \times 10^{-15} \text{ meters}$$

(This is the radius such that the rest mass of the electron is attributed to the energy necessary to assemble the charge e on a sphere of radius r_{class} : $mc^2 = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r_{class}}$)

$$\Rightarrow \quad 3r^2 \frac{dr}{dt} = -4c\, r_{class}^2 \quad \Rightarrow \quad \frac{d}{dt}(r^3) = -4c\, r_{class}^2 .$$

Solution: $r^3(t) = r^3(t=0) - 4ct\, r_{class}^2 = r_0^3 - 4ct\, r_{class}^2$

Take the initial radius to be the Bohr radius

$$r(t=0) = a_0 = 5\times 10^{-10} \text{ m} = 5 \times 10^{-11} \text{ m}$$

Decay time: $r(t_{decay}) = 0$ (electron crashes into nucleus)

$$\Rightarrow t_{decay} = \frac{a_0^3}{4c\, r_{class}^2} = \frac{(5\times 10^{-11} \text{ m})^3}{4\,(3\times 10^{10} \text{ m/s})(2.8\times 10^{-15} \text{ m})^2}$$

$$= 1.33 \times 10^{-13} \text{ s}$$

$$\Rightarrow \boxed{t_{decay} = 13.3 \text{ ps}}$$

So, classical atoms can't exist!

Quantum mechanics saves the day.

(b) Generally we need two things:

$$E(r) = \begin{pmatrix} \text{energy of circular} \\ \text{orbit at radius } r \end{pmatrix}$$

$$P_{rad}(r) = \begin{pmatrix} \text{time-average power going} \\ \text{into dipole radiation for} \\ \text{orbit at radius } r \end{pmatrix}$$

Energy balance:

$$P_{rad}(r) = -\frac{dE}{dt} = -\frac{dE}{dr}\frac{dr}{dt}$$

$$\Longrightarrow \frac{dr}{dt} = -\frac{P_{rad}(r)}{dE/dr} \iff \boxed{dt = -\frac{dE/dr}{P_{rad}(r)}\, dr}$$

Coulomb potential

$$V(r) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r}$$

$$F_{centrifugal} = +\frac{mv^2}{r}$$

$$F_{Coulomb} = -\frac{dV}{dr} = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}$$

Force balance in radial direction

$$\frac{mv^2}{r} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{r^2}$$

$$\Longrightarrow v^2 = \frac{e^2}{4\pi\epsilon_0 m}\frac{1}{r} = r_{class}\frac{c^2}{r}$$

$$\omega^2 = \left(\frac{v}{r}\right)^2 = r_{class}\frac{c^2}{r^3}$$

ω is angular velocity or frequency

Quadratic potential

$$V(r) = \frac{1}{2}kr^2 \longleftarrow$$ Isotropic spring w/ spring constant k

$$F_{centrifugal} = \frac{mv^2}{r}$$

$$F_{potential} = -\frac{dV}{dr} = -kr$$

Force balance in radial direction:

$$\frac{mv^2}{r} = kr$$

$$\Longrightarrow v^2 = \frac{k}{m}r^2$$

$$\omega^2 = \frac{v^2}{r^2} = \frac{k}{m} \longleftarrow$$ constant indep of r

This gives

$$E(r) = \frac{1}{2}mv^2 + V(r)$$

$$= \frac{1}{2}\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} - \frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

$$= -\frac{1}{2}\frac{e^2}{4\pi\epsilon_0}\frac{1}{r}$$

$$E(r) = -\frac{1}{2}mc^2\frac{r_{class}}{r} \propto \frac{1}{r}$$

$$P_{rad}(r) = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{p_0^2\omega^4}{c^3}$$

$$p_0 = -er, \quad \omega^2 = r_{class}\frac{c^2}{r^3}$$

$$P_{rad}(r) = \frac{2}{3}\frac{e^2}{4\pi\epsilon_0 c^2} r^2 r_{class}^2 \frac{c^{\cancel{4}\,3}}{r^6}\frac{1}{\cancel{c}}$$

$$P_{rad}(r) = \frac{2}{3}mc^3\frac{r_{class}^3}{r^4} \propto \frac{1}{r^4}$$

$$\frac{dE}{dr} = \frac{1}{2}mc^2\frac{r_{class}}{r^2} \propto \frac{1}{r^2}$$

P_{rad} increases much faster than $\frac{dE}{dr}$ as r decreases.

$$dt = -\frac{dE/dr}{P_{rad}(r)}\,dr$$

$$= -\frac{\frac{1}{2}mc^2 r_{class}/r^2}{\frac{2}{3}mc^3 r_{class}^3/r^4}\,dr$$

$$= -\frac{3}{4}\frac{r^2}{r_{class}^2}\frac{dr}{c}$$

$$E(r) = \frac{1}{2}mv^2 + V(r)$$

$$= \frac{1}{2}kr^2 + \frac{1}{2}kr^2$$

$$E(r) = kr^2 = m\omega^2 r^2$$

Kinetic and potential energies are the same for a harmonic oscillator

$$P_{rad}(r) = \frac{1}{4\pi\epsilon_0}\frac{2}{3}\frac{p_0^2\omega^4}{c^3}$$

$$p_0 = -er, \quad \omega^2 = k/m$$

$$P_{rad}(r) = \frac{2}{3}\frac{e^2}{4\pi\epsilon_0 c^2} r^2 \frac{\omega^4}{c}$$

$$P_{rad}(r) = \frac{2}{3}\frac{m}{c} r_{class} r^2 \omega^4$$

$$\frac{dE}{dr} = 2m\omega^2 r$$

$P_{rad}(r)$ decreases faster than dE/dr as r decreases

$$dt = -\frac{dE/dr}{P_{rad}(r)}\,dr$$

$$= -\frac{2m\omega^2 r}{\frac{2}{3}\frac{m}{c} r_{class} r^2 \omega^4}$$

$$= -3\frac{c}{r_{class}\omega^2}\frac{dr}{r}$$

Integrate:

$$t = -\frac{1}{4}\frac{1}{c r_{class}^2}\underbrace{\int_{r_0}^{r} 3r^2 dr}_{r^3 - r_0^3}$$

$$4ctr_{class}^2 = r_0^3 - r^3$$

$$\boxed{r^3 = r_0^3 - 4ctr_{class}^2}$$

$r=0$ at $t = \dfrac{r_0^3}{4cr_{class}^2}$

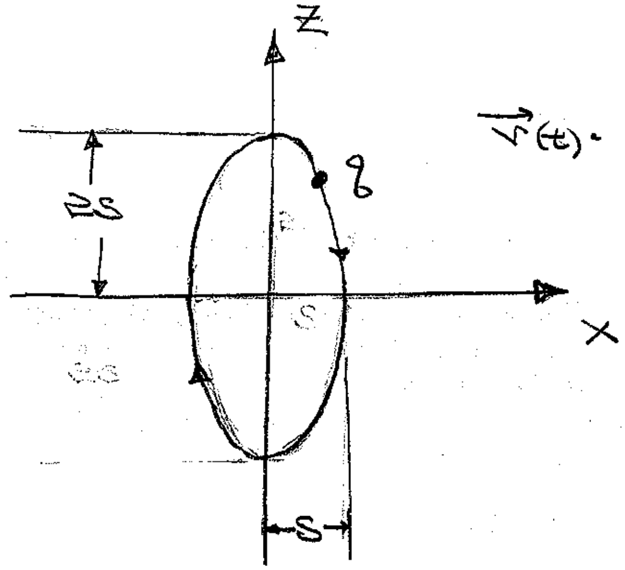
Integrate:

$$t = -3\frac{c}{\omega^2 r_{class}}\underbrace{\int_{r_0}^{r}\frac{dr}{r}}_{\ln\left(\frac{r}{r_0}\right)}$$

$$\therefore \boxed{r = r_0 \exp\left(-\frac{\omega^2 r_{class}}{3c}t\right)}$$

Orbit decays exponentially; ~~it~~ takes forever to get to $r=0$

11.3.



$$\vec{r}(t) = 2s\cos\omega t\,\hat{e}_z + \sin\omega t\,\hat{e}_x$$

(a) $$\boxed{\vec{p}(t) = q\vec{r}(t) = \overbrace{qs}^{\equiv p_0}(2\cos\omega t\,\hat{e}_x + \sin\omega t\,\hat{e}_x)}$$

$$= \mathrm{Re}\Big(\underbrace{p_0(2\hat{e}_z + i\hat{e}_x)}_{\tilde{\vec{p}}}\,e^{-i\omega t}\Big)$$

$$\boxed{\tilde{\vec{p}} = p_0(2\hat{e}_z + i\hat{e}_x)}$$

(b) From notes, $\vec{E} = \mathrm{Re}(\tilde{\vec{E}}e^{-i\omega t})$, where

$$\tilde{\vec{E}} = \frac{\mu_0\omega^2}{4\pi}\frac{\tilde{\vec{p}}_T e^{ikr}}{r}, \qquad \tilde{\vec{p}}_T \equiv \tilde{\vec{p}} - \hat{e}_r(\hat{e}_r\cdot\tilde{\vec{p}})$$

We need to calculate $\tilde{\vec{p}}_T$:

$$\tilde{\vec{p}}_T = \tilde{\vec{p}} - \hat{e}_r\underbrace{(\hat{e}_r\cdot\tilde{\vec{p}})}$$

$$p_0(2\hat{e}_z\cdot\hat{e}_r + i\hat{e}_x\cdot\hat{e}_r) = p_0(2\cos\theta + i\sin\theta\cos\phi)$$

$$= p_0\Big(2\hat{e}_z + i\hat{e}_x - \hat{e}_r(2\cos\theta + i\sin\theta\cos\phi)\Big)$$

Write $\hat{e}_z = \underbrace{(\hat{e}_z\cdot\hat{e}_r)}_{\cos\theta}\hat{e}_r + \underbrace{(\hat{e}_z\cdot\hat{e}_\theta)}_{-\sin\theta}\hat{e}_\theta + \underbrace{(\hat{e}_z\cdot\hat{e}_\phi)}_{0}\hat{e}_\phi$

(from text book)

$$= \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$$

$$\hat{e}_x = \underbrace{(\hat{e}_x\cdot\hat{e}_r)}_{\sin\theta\cos\phi}\hat{e}_r + \underbrace{(\hat{e}_x\cdot\hat{e}_\theta)}_{+\cos\theta\cos\phi}\hat{e}_\theta + \underbrace{(\hat{e}_x\cdot\hat{e}_\phi)}_{-\sin\phi}\hat{e}_\phi$$

$$= \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi$$

$$\vec{\tilde{p}}_T = p_0\big(2\cos\theta\,\hat{e}_r - 2\sin\theta\,\hat{e}_\theta$$
$$+ i\sin\theta\cos\phi\,\hat{e}_r + i\cos\theta\cos\phi\,\hat{e}_\theta - i\sin\phi\,\hat{e}_\phi$$
$$- \hat{e}_r(2\cos\theta + i\sin\theta\cos\phi)\big)$$

$$= p_0\big((-2\sin\theta + i\cos\theta\cos\phi)\hat{e}_\theta - i\sin\phi\,\hat{e}_\phi\big)$$

$$\boxed{\vec{\tilde{E}} = \frac{\mu_0\omega^2 p_0}{4\pi}\Big((-2\sin\theta + i\cos\theta\cos\phi)\hat{e}_\theta - i\sin\phi\,\hat{e}_\phi\Big)\frac{e^{ikr}}{r}}$$

(c) Along the positive x axis, $r = x$ and $\hat{e}_r = \hat{e}_x$, so

$$\vec{\tilde{p}}_T = \vec{\tilde{p}} - \hat{e}_x(\hat{e}_x\cdot\vec{\tilde{p}}) = 2p_0\hat{e}_z.$$

Or use $\theta = \frac{\pi}{2}$, $\phi = 0$, $r = x$, $\hat{e}_\theta = -\hat{e}_z$, and $\hat{e}_\phi = \hat{e}_y$ in $\vec{\tilde{E}}$ of part (b).

Thus

$$\vec{\tilde{E}} = \frac{\mu_0\omega^2 p_0}{2\pi}\hat{e}_z\frac{e^{ikx}}{x}$$

$$\vec{E} = \mathrm{Re}\big(\vec{\tilde{E}}e^{-i\omega t}\big) = \boxed{\hat{e}_z\frac{\mu_0\omega^2 p_0}{2\pi}\frac{\cos(kx-\omega t)}{x} = \vec{E}}$$

$\boxed{\text{$\vec{E}$ is linearly polarized along $\hat{e}_z$ because the projected dipole is a charge oscillating along the z axis.}}$

(d) Along the positive z axis, $r = z$ and $\hat{e}_r = \hat{e}_z$, so

$$\vec{\tilde{p}}_T = \vec{\tilde{p}} - \hat{e}_z(\hat{e}_z \cdot \vec{\tilde{p}}) = i p_0 \hat{e}_x$$

Thus

$$\vec{\tilde{E}} = \frac{\mu_0 \omega^2 p_0}{4\pi} \hat{e}_x \frac{i e^{ikz}}{z}$$

$$\vec{E} = \mathrm{Re}\left(\vec{\tilde{E}} e^{-i\omega t}\right) = \boxed{-\hat{e}_x \frac{\mu_0 \omega^2 p_0}{4\pi} \frac{\sin(kz-\omega t)}{z} = \vec{E}}$$

$\boxed{\text{$\vec{E}$ is linearly polarized along $\hat{e}_x$ because the projected dipole is a charge oscillating along the x axis.}}$

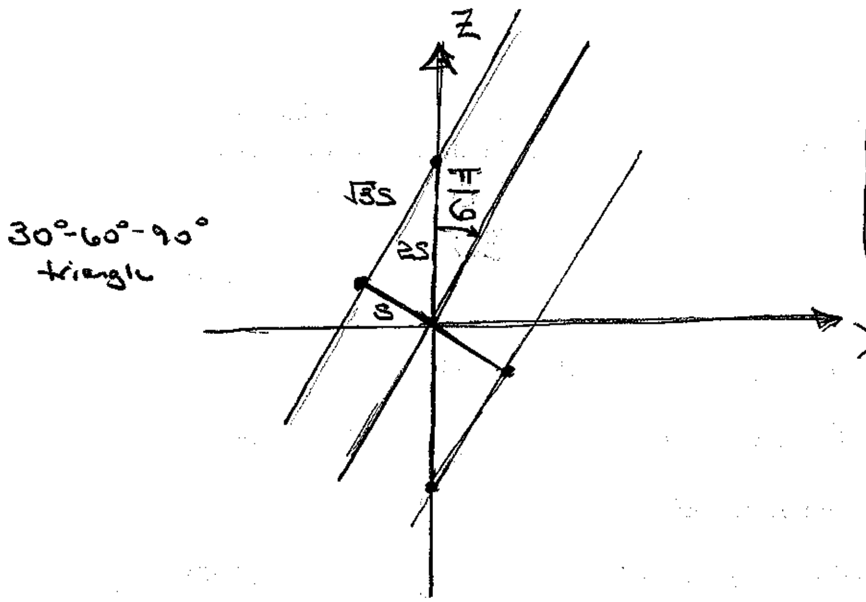
(e) $\phi = \pi/2$: $\vec{\tilde{E}} = \frac{\mu_0 \omega^2 p_0}{4\pi}\left(-2\sin\theta\, \hat{e}_\theta - i\hat{e}_\phi\right)\frac{e^{ikr}}{r}$

$\cos\phi = 0$, $\sin\phi = 1$ $\Longleftrightarrow$ x-z plane with $y > 0$

The field is circulary polarized when the θ and ϕ oscillations have the same size and are $90°$ out of phase, i.e., $2\sin\theta = 1$ or

$$\boxed{\sin\theta = \frac{1}{2} \Longleftrightarrow \theta = \frac{\pi}{6} = 30°}$$

(4)



The projected z oscillation has amplitude s, the same as the x oscillation, so the projected dipole moves in a circle, producing circular polarization.

4)

As we are only given the difference in time between the two flashes, we first decide to synchronize our watch with the first flash; i.e., we assign the time we ascribe to its flash as $t = 0$. Therefore, we may then set out the spacetime coordinats for the two events:

$$\text{large flash: } x_1 = +1650 \;\text{ m}, ct_1 = 0 \text{ m} ,$$

$$\text{small flash: } x_2 = +1650 - 850 \;\text{ m} = 800 \text{ m}, ct_2 = (300 \text{ m}/\mu \text{ s})(5\mu \text{ s}) = 1500 \text{ m} .$$

We then determine the value of the invariant interval between the two events:

$$(\Delta s)^2 \equiv (\Delta x)^2 - (\Delta ct)^2 = (1650 - 800)^2 - (0 - 1500)^2 = -1527500 = -(1235.92 \text{ m})^2 .$$

As this interval is negative, we ascribe a proper-time difference between the two events:

$$\Delta\tau = \sqrt{-(\Delta s/c)^2} = \pm 1235.92 \text{ m}/300 \text{ m}/\mu \text{ s} = \pm 4.12\mu \text{ s} ,$$

where we have not yet determined the correct sign for this time difference. However, in the frame where this $\Delta\tau$ is the time difference between the two events, we see that they have $\Delta x' = 0$, so that the observer in question must have actually been at both events. This means that his velocity had to be

$$v = \Delta x/\Delta t = \pm(850 \text{ m})/(1500 \text{ m}/c) = \pm 0.5667c ,$$

which corresponds to a relativistic factor of $\gamma = 1.2137$. To (finally) determine the sign, we consider the Lorentz transformations between the two frames:

$$x_1' = \gamma(x_1 - vt_1) = 1.2137[1650 - 0] = 2002.6 \text{ m} ,$$

$$x_2' = \gamma(x_2 - vt_2) = 1.2137[800 - (\pm 0.5667)(1500)] = \begin{cases} -60.7 \text{ m} , & + \text{ sign,} \\ 2002.6 \text{ m} , & - \text{ sign.} \end{cases}$$

Since our observer needs to have been at both events, i.e., they should both the same value of x', we see that it is the minus sign that is correct; i.e., $v = -0.5667c$. Lastly, we compute now the time differences, to determine which one occurred first, from this traveler's point of view:

$$ct_1' = \gamma(ct_1 - (v/c)x_1) = 1.2137[0 + .5667(1650)] = 1134.8 \text{ m} ,$$

$$ct_2' = \gamma(ct_2 - (v/c)x_2) = 1.2137[1500 + .5667(800)] = 2370.1 \text{ m} .$$

This tells us that in the moving observer's frame, the second flash occurred later; i.e., they occurred in the same sequence in both frames.

Since our observer needs to have been at both events, i.e., they should both the same value of x', we see that it is the minus sign that is correct; i.e., $v = -0.5667c$. Lastly, we compute now the time differences, to determine which one occurred first, from this traveler's point of view:

$$ct'_1 = \gamma(ct_1 - (v/c)x_1) = 1.2137[0 + .5667(1650)] = 1134.8 \text{ m},$$
$$ct'_2 = \gamma(ct_2 - (v/c)x_2) = 1.2137[1500 + .5667(800)] = 2370.1 \text{ m}.$$

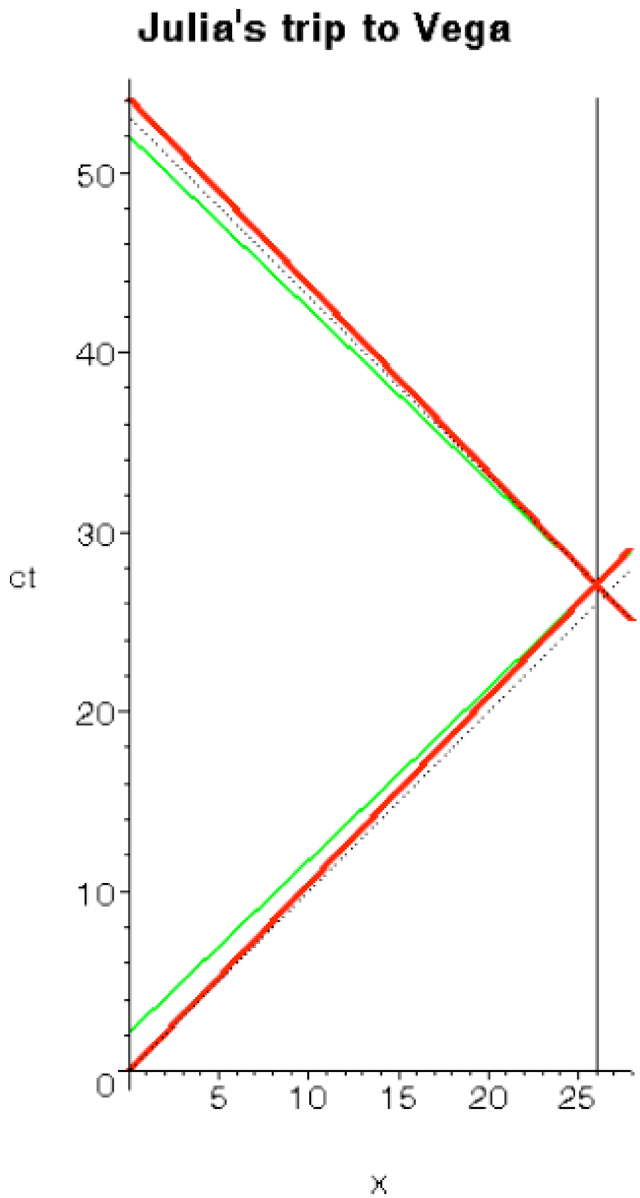
This tells us that in the moving observer's frame, the second flash occurred later; i.e., they occurred in the same sequence in both frames.

5)

We begin by calculating the relativistic factor for Julia's trip(s):

$$\gamma = 1/\sqrt{1 - .96^2} = 3.5714\,.$$

I then believe it is best to go ahead and put the solution to part (e), i.e., the spacetime diagram right here, to help with the visualization of the remainder of the problem; of course it was necessary to have some of the numbers calculated below in order to produce this diagram.



The figure to the left shows just the worldlines of the earth and the star Vega (in black), Julia's outward-bound trip and her return trip (in red), as well as two of her lines of simultaneity with her landing on Vega, one for the outward worldline and one for the inward worldline, both in green, and a couple of reference light rays in dotted black.

The radio waves are not shown on this figure, just for clarity. They are included in the next one, shown at the end of this solution.

a. For Julia's outward-bound trip, Earth measures the trip to cover 26 lightyears and to be traversed at speed $v = 0.96c$.

i.) Therefore Earth's clocks record the time T for the trip:

$$T = L/v = 26c \text{ years}/(.96c) = 27.083 \text{ years}.$$

ii.) From Earth's point of view, Julia's time runs more slowly by the relativistic factor, so that

$$T' = T/\gamma = 27.083/3.5714 = 7.583 \text{ years}.$$

It is worth checking that this number is the same as what Julia would have said herself: She believes Vega is only the distance $L' = L/\gamma$ away from her, and that it is approaching at the speed $v = 0.96c$; therefore, it should take a time $T' = L'/v = L/\gamma/v = T/\gamma$ to do that, which is the answer we got above.

iii.) Now, Earth sends out a radiogram each year, beginning 1 year after Julia left. Each of them takes a further 26 years to arrive at Vega. Therefore, she should have received one and only one such radiogram, while outward bound, somewhat before she actually reached Vega. To be more sure about this question, we think about the two events that correspond to the sending and receiving of a radiogram. In Earth coordinates, the events of sending the n-th radiogram correspond to $x_n = 0$ and $t_n = n$, measured in years. The events of reception correspond to the intersection of the straight line representing the path of the light ray, moving at 45° on our Minkowski diagram, with Julia' straight line, moving at speed v. We model those two straight lines as

$$\text{Julia's world line: } ct = (c/v)x \ ,$$

$$\text{the } n\text{-th light ray: } ct = x + nc \ .$$

These intersection events then have the following coordinates:

$$\text{on Earth:} \quad x_n = \frac{nv}{1 - v/c} \ , \ ct_n = \frac{nc}{1 - v/c} \ ;$$

$$\text{on Julia's ship:} \quad \begin{cases} x'_n = \gamma[x_n - (v/c)t_n] = \dfrac{\gamma}{1 - v/c}[nv - (v/c)nc] = 0 \ , \\ ct'_n = \gamma[ct_n - (v/c)x_n] = \dfrac{n\gamma c}{1 - v/c}[1 - v^2/c^2] = nc\sqrt{\dfrac{1 + v/c}{1 - v/c}} \\ \qquad = n\sqrt{\dfrac{1.96}{.04}} = \sqrt{49}\, n = 7n \text{ years. .} \end{cases}$$

Since she did in fact arrive in 7.583 of her years, we see that indeed **she only received 1 radiogram from Earth prior to the beginning of her return trip.**

iv.) Since the two views are entirely consistent, one with respect to the other, since each one thinks they are at rest and the other one is moving, the same calculation above, switching primes and unprimes, gives us the desired equations for the radiograms that Julia sends out. She sends them out at $x'_n = 0$ and $t'_n = n$, measured in years as usual. Therefore we find that the intersections with the worldline of the Earth, namely $ct = -x(c/v)$, gives us

$$\text{on board ship:} \qquad x'_m = \frac{-mv}{1 - v/c}\,,\; ct'_m = \frac{mc}{1 - v/c}\,;$$

$$\text{on Earth:} \qquad x_m = 0\,, \quad t_m = m\sqrt{\frac{1 + v/c}{1 - v/c}} = 7m\,.$$

So radiograms that she sent out on the outward trip would be received, on Earth, every 7 years. However, since she only travelled, as she measured it, for 7.583 years, this means that Earth would receive 7 of those, every 7 years, so that the last one would be received after 49 years. Since she arrived on Vega, by Earth measurements, after 27.08 years, the answer to the question is that **during her travel outward, Earth received just 3 of her radiograms**, the remaining 4 to be received while she is actually already returning.

v.) Lastly, while she spent only 7.583 years travelling, by her clocks, she perceived Earth as moving away from her with speed $v = 0.96c$ and relativistic factor $\gamma = 3.571$, so that by her clocks, when she arrived on Vega, the time on Earth was given by

$$T'' = T'\gamma = (7.583 \text{ years})/3.571 = 2.123 \text{ years.}$$

b. The return trip is in many ways the same as the outgoing trip since the magnitude of the speed, and therefore the relativistic factor is the same.

i.) The traveler believes that Earth is now as far away as Vega once was, and is now traveling at the same speed toward her. Therefore, the trip requires

$$T' = 7.583 \text{ years.}$$

ii.) Clearly the person on Earth also thinks she must return the same distance at the same speed, so that they say that

$$T = \gamma T' = 27.083 \text{ years.}$$

iii.) As the people on Earth will, by the time she returns, after a total of 54.167 Earth years, have sent out 54 radiograms, **she must receive 53 of them on the return trip.**

iv.) As she has sent out, by the time she returns, after a total of 15.167 years, as she measures time, have sent out 15 radiograms, and they have alrady received 3 before she turned around, **they will receive 12 more before her final return home.** One could verify all this by working out details of frequencies between radiogram arrivals. If this were done, it would be found that the relations are the same as before, except that now one must replace v by $-v$, which results in a time period between receptions of one seventh of one year, instead of 7 years.

v.) Obviously she, again, thinks that only 2.123 years have elapsed on Earth during her return trip, since she views Earth as travelling toward her at $v = 0.96c$.

c. Since, on Earth, 54.167 years have elapsed between her leaving and her returning, and she measures that only 4.247 years have elapsed during her two travel times, it must be that during her (minuscule) turn-around "jump" between frames,
some 49.92 years elapsed on Earth.

d. To verify this in a slightly more pedantic way, let us attempt to consider her two lines of simulataneity. On her trip outward, her lines of simultaneity are straight lines parallel to the original one, as she left. That original line was just $ct = (v/c)x$; therefore, the one parallel to that which goes through the event of her arrival is

$$ct - ct_A = \frac{v}{c}(x - x_A) \quad \text{is the line of simultaneity with her arrival.}$$

She therefore inserts $x = 0$ into this equation to determine the time, t_1 on Earth, in Earth's coordinates, that she views as simultaneous with her arrival on Vega:

$$ct_1 = ct_A - \frac{v}{c}x_A = ct_A\left[1 - \left(\frac{v}{c}\right)^2\right] = ct_A/\gamma^2 = ct'_A/\gamma = 2.123 \text{ years.}$$

In different words, she views the people on Earth as only having aged 2.123 years during the time she watched them leave her frame at a fast speed and while she aged 7.583 years.
On the trip back, her worldline may be written as

$$ct - ct_A = -\frac{c}{v}(x - x_A)\,,$$

so that her line of simultaneity with that event, in her newly-established frame, is given by

$$ct - ct_A = -\frac{v}{c}(x - x_A)\,.$$

The intersection of that line with Earth is when $x = 0$, giving us the time t_2, simultaneous with her departure, but on Earth, and in Earth's coordinates, as

$$ct_2 = ct_{_A} + \frac{v}{c}x_{_A} \ .$$

Therefore, the time difference, on Earth, between her measurement of her arrival on Vega, in an outgoing frame, and her measurement of her departure from Vega, in an incoming frame is given by $t_2 - t_1$, which is

$$ct_2 - ct_1 = [ct_{_A} + \frac{v}{c}x_{_A}] - [ct_{_A} - \frac{v}{c}x_{_A}] = 2(v/c)x_{_A} = 2(.96)(26c) \text{ years} = 54.16 \text{ lightyears.}$$

This is of course an enormous amount, for her instantaneously-accomplished shift of frames—i.e., a tremendous acceleration. Nonetheless, when this amount is added to the time she measured to have passed on Earth during her trip outward, i.e., 2.123 years, plus the same amount of time, as she measures it, for her return trip, it adds up to the total time that passed on earth while she was gone. This tells us that this small period of acceleration, **i.e., not being in an inertial frame, caused an enormous disparity in the observations made by the two persons.**