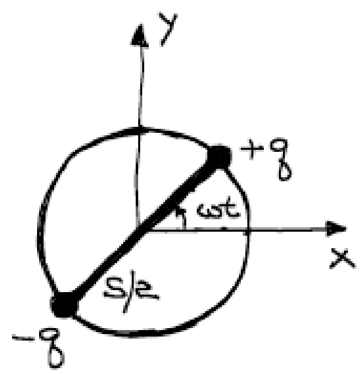
Homework #9 *due date is Tuesday 11/18/2013* ***in class***
(see syllabus about late HWs)
(Note that all Griffiths Problems are from 4th Ed.)

1) Problem on calculating the radiation from an oscillating dipole. Consider an electric dipole rotating in the x-y plane. The motion of this dipole is a superposition of two linearly oscillating dipoles that are 90° out of phase with one another:

$$\mathbf{p}(t) = \hat{\mathbf{e}}_x p_0 \cos\omega t + \hat{\mathbf{e}}_y p_0 \sin\omega t = \mathrm{Re}(\tilde{\mathbf{p}} e^{-i\omega t}) \,.$$

Here $p_0 = qs$, and the complex amplitude of the dipole moment is

$$\tilde{\mathbf{p}} = p_0(\hat{\mathbf{e}}_x + i\hat{\mathbf{e}}_y) \,.$$



(a) *Show* that the vector phasor of the electric field in the far (radiation) zone, expressed in spherical coördinates, is

$$\tilde{\mathbf{E}} = \frac{k^2 p_0}{4\pi\epsilon_0}(\cos\theta\,\hat{\mathbf{e}}_\theta + i\hat{\mathbf{e}}_\phi)\frac{e^{i(kr+\phi)}}{r} \,.$$

Explain the presence of the azimuthal angle ϕ in the phase of this wave. (Hint: Treat the problem as the superposition of the fields associated with the two linearly oscillating dipoles, or use the general results in the lecture notes. It would be best to do both.)

(b) *Show* that on the positive x, y, and z axes, the (real) electric fields are given by

Positive x axis ($\theta = \pi/2$, $\phi = 0$): $$\mathbf{E}(\mathbf{r},t) = \frac{k^2 p_0}{4\pi\epsilon_0}\frac{\sin(\omega t - kx)}{x}\hat{\mathbf{e}}_y \,,$$

Positive y axis ($\theta = \pi/2$, $\phi = \pi/2$): $$\mathbf{E}(\mathbf{r},t) = \frac{k^2 p_0}{4\pi\epsilon_0}\frac{\cos(\omega t - ky)}{y}\hat{\mathbf{e}}_x \,,$$

Positive z axis ($\theta = 0$): $$\mathbf{E}(\mathbf{r},t) = \frac{k^2 p_0}{4\pi\epsilon_0}\frac{\cos(\omega t - kz)\hat{\mathbf{e}}_x + \sin(\omega t - kz)\hat{\mathbf{e}}_y}{z} \,.$$

Remember that the relation between the spherical unit vectors and the Cartesian unit vectors depends on position. What is the polarization of the electric field associated with these three waves (linear, circular, etc.)? *Explain* why these polarizations are what you would expect.

(c) *Show* that the average power radiated per solid angle is

$$\frac{dP}{d\Omega} = r^2 \langle \mathbf{S} \rangle \cdot \hat{\mathbf{e}}_r = \frac{1}{4\pi\epsilon_0} \frac{ck^4 p_0^2}{8\pi} (1 + \cos^2\theta) \, .$$

Remember that $\langle \mathbf{S} \rangle = (c\epsilon_0/2)(\tilde{\mathbf{E}} \cdot \tilde{\mathbf{E}}^*)\hat{\mathbf{e}}_r$. *Sketch* the power per solid angle as a function of θ.

(d) *Show* that the total power radiated into all 4π steradians is twice that for a linearly oscillating dipole, i.e.,

$$P = \int d\Omega \, \frac{dP}{d\Omega} = \frac{1}{4\pi\epsilon_0} \frac{2}{3} ck^4 p_0^2 \, .$$

Where does the factor of two come from? Why not a factor of four, since fields usually add, but energy usually doesn't because of interference?

2) The death of the classical atom. In the early part of the 20th century, after Thomson's discovery of the electron and Rutherford's discovery of the atomic nucleus, a classical model of the atom was proposed. In this model electrons orbit the nucleus, in the same way that the planets orbit the sun, only in the atom the binding force is the Coulomb attraction rather than the gravitational force. There is a problem with this model: the electron-nucleus system is a rotating dipole and thus will radiate electromagnetic waves that carry away energy! As an electron loses energy, the radius of its orbit will decrease, and eventually it will spiral into the nucleus.

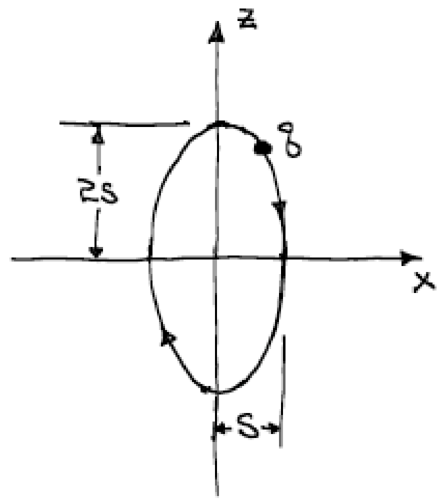
(a) *Calculate* the time it would take for a hydrogen atom, modeled as an electron in a circular orbit about a proton, to lose all its energy to dipole radiation. First *find* how the radius of the orbit decays from an initial radius r_0 (you will need to use the results of the preceding problem), and then taking the initial radius to be the Bohr radius, i.e., $r_0 = a_0 = 0.5\,\text{Å}$, *find* the time it takes for the electron to crash into the nucleus.

The paradox that a classical atom is unstable because of emission of electromagnetic radiation led Bohr to make his gigantic leap into the quantum theory by postulating that the orbits are by definition *stationary states* that do not radiate.

(b) How would the results of part (a) change if the electron were bound to the proton by a quadratic potential $V(r) = \frac{1}{2}kr^2$?

3) Consider a charge q which moves on the elliptical trajectory given by:

$$\mathbf{r}(t) = 2s\cos\omega t\,\hat{\mathbf{e}}_z + s\sin\omega t\,\hat{\mathbf{e}}_x \ .$$



(a) *Give* the time-dependent dipole moment $\mathbf{p}(t)$ and the dipole phasor $\tilde{\mathbf{p}}$, where $\mathbf{p}(t) = \mathrm{Re}(\tilde{\mathbf{p}}e^{-i\omega t})$.

(b) *Show* that the vector phasor of the electric field in the far (radiation) zone, expressed in spherical coördinates, is

$$\tilde{\mathbf{E}} = \frac{\mu_0\omega^2 p_0}{4\pi}\Big((-2\sin\theta + i\cos\theta\cos\phi)\hat{\mathbf{e}}_\theta - i\sin\phi\,\hat{\mathbf{e}}_\phi\Big)\frac{e^{ikr}}{r} \ ,$$

where $p_0 = qs$.

(c) *Find* the (real) electric field along the positive x axis, and *express* it in Cartesian coördinates. How is the electric field polarized? *Interpret* your answer.

(d) *Find* the (real) electric field along the positive z axis, and *express* it in Cartesian coördinates. How is the electric field polarized? *Interpret* your answer. [Hint: In doing this part, it might be wise not to use directly the field found in part (b), but rather to use an intermediate step along the road to the field of part (b).]

(e) For $\phi = \pi/2$, *find* the value of θ between 0 and $\pi/2$ such that the electric field is circularly polarized. *Interpret* your answer.

A Listing of Some Basic Concepts from Special Relativity:

1. The two basic postulates:
   a.) any inertial frame acceptable,
   b.) and the constancy of the speed of light.
2. 4-dimensional geometry, instead of 3 plus 1
   a.) frame dependence of simultaneity
   b.) time dilation and space contraction, when moving between inertial frames
   c.) timelike coordinates are different from spacelike/spatial coordinates
   d.) the Minkowski metric generates the invariant "interval," the same as measured by all inertial observers
3. Lorentz transformations, review them in at least one spatial and one temporal direction
   a.) there is a mixing of z and t when moving between inertial frames
   b.) review velocity-addition formulae [1 spatial dimension only]

FOR PRACTICE, NOT TO TURN IN: Griffiths 12.7, 12.8
Then do the following:
4)

2. An observer $\mathcal{O}$ sees a large flash of light at a distance of 1650 meters from his position. Five microseconds later he sees a small flash of light 850 meters closer to him, directly along the line to the (earlier) position of the larger flash.

a. Show that there is a different observer that would have recorded these two flashes as having occurred at the same place. Find the velocity of that observer, as measured by $\mathcal{O}$, and the time between the flashes as measured by this new observer.

b. Which flash does this new observer say occurred first?

5) Mostly Xtra Credit

A space traveler, Julia, takes oÆ from Earth and moves at speed v = 0.96c toward the star Vega, which is 26.0 lightyears away, as measured by people on Earth. (Assume that the star Vega is at rest with respect to the Earth.) The traveler and the people on Earth both agree to send each other a radio-gram each year! [**Note** that it is probably useful to refer to the observers on Earth as O, while observations made by Julia on her way to Vega we can refer to as measurements made by O'.] **Also** be sure to start constructing the spacetime diagram requested in part (e) as you move along calculating the various parts of the problem below. It should help you, especially with respect to the question(s) in parts (c) and (d).

a. When the traveler reaches Vega:
   i.) How much time will have elapsed by Earth clocks?
   ii.) How much time will have elapsed by the traveler's clocks?
   iii.) How many radio-grams have been received by the traveler?
   iv.) How many radio-grams will have been received on Earth?

v.) How much time does the traveler's reference frame measure to have elapsed on Earth? Do note that this is the tricky one, since Julia measures herself to be at rest, and the Earth as moving away from her, while Vega is travelling toward her. Therefore, the question is asking for the value of *ct* on Earth for the event that she believes is simultaneous with her landing on Vega.

b. The very instant the traveler arrives at Vega, she \jumps onto" a ship headed back toward Earth, moving at the same speed as she was before, but of course in the opposite direction. Now that we have a third reference frame involved, please refer to this frame as O''.

i.) How much time is required for the return trip, as measured by the traveler?

ii.) and how much time is required as measured by the persons on Earth?

iii.) How many radio-grams will be received by the traveler during this return trip?

iv.) How many radio-grams will be received on Earth?

v.) How much time does the traveler's new reference frame measure to elapse on Earth during the return trip?

c. How much time passed on Earth, as measured by their clocks, during the change that Julia made from her Ørst reference frame to her second?

d. Justify the answer just above by determining equations for the two lines of simultaneity that Julia's two reference frames possess, while she is on Vega; the one frame is the one while she was travelling outward, and the other is the one while she was travelling backward. (By equations I mean the equation for some straight line in the $x; ct$-plane.) Then Ønd the two intersections of these two lines with the ct-axis, i.e., the Earth-bound person's world line, and show that the time between those two intersections is indeed the answer to part (c).

e. Please draw a reasonably-accurate, and neat and clean, spacetime diagram for this problem, which should show the Earth's worldline and also Vega's worldline, Julia's two diÆerent world lines, i.e., the outward-bound portion of the trip, and the return portion of the trip, and also her two diÆerent lines of simultaneity with respect to her landing on Vega, i.e., the lines through that event for the two diÆerent reference frames in which she has found herself.

A somewhat more detailed way of saying the same thing is the following:

i.) lines for O: the diagram should show the x- and ct-axes and a line parallel to the ct-axis through Vega's location, i.e., Vega's worldline;

ii.) lines for O': the ct'-axis and x'-axis, and a line parallel to the x'-axis which passes through the event of her landing there, i.e., her line of simultaneity with the event of her landing;

iii.) lines for O^{∞}: one may either choose the origin for frame O^{∞} to go through the same origin as all the others, or | what is simpler and easier | to choose its origin as the event of the landing. I don't actually care which. Let us assume, in what's written below, that you have chosen the origin as the landing event; in that case please draw the ct^{∞}- and x^{∞}-axes, through that origin, and extend the x^{∞}-axis backwards until it intersects the worldline of the earth. If, instead, you choose the same origin as the others for O^{∞}, then the lines in question should be parallel to those axes and passing through the landing event.

iv.) As well, please put some lines showing some of the radio-grams that were sent. As the diagram would become much too complicated if all of the radio-grams were shown, show only, say 4 such lines for radio-grams sent from Earth to Julia, some sent before she arrived and some afterward, and also show 4 sent by Julia to Earth, 2 sent before she arrived, and 2 after she is on her return voyage, in frame O^{∞}.