Homework #8 Solutions

1)

1. (a) Unless the same $x,y$ dependence holds for both traveling components of a mode, the modefields will not obey the required B.C.s at the cross-sectional boundary. In other words, for a given TE (or TM) mode,

$$B_z = B_z^{(+)}(x,y)\, e^{ik^{(+)}z - i\omega t} + B_z^{(-)}(x,y)\, e^{-ik^{(-)}z - i\omega t}$$

cannot obey (for all $z$) the BC obeyed by $B_z^{(+)}(x,y)$, unless $B_z^{(-)}(x,y)$ is simply proportional to it so the $(x,y)$ dependence factorizes from the $(z,t)$ dependences. That would require $k^{(+)}$ and $k^{(-)}$ to have the same magnitude, $k^{(+)2} = k^{(-)2} \equiv k^2$, since

$$\left(\nabla_T^2 + k_\pm^2 - \omega^2 \epsilon\mu\right) B_z^{(\pm)}(x,y) = 0$$

and $B_z^{(-)} \propto B_z^{(+)}$

(b) The most general solution is what's written above for $B_z$ and can similarly be written for the remaining EM field components, i.e., since $B_z^{(+)} \propto B_z^{(-)}$

$$\boxed{\tilde{B}_z = B_z(x,y)\left(\underset{\sim}{a}\, e^{ikz} + \underset{\sim}{b}\, e^{-ikz}\right) e^{-i\omega t},}$$

$\underset{\sim}{a}, \underset{\sim}{b}$ : arbitrary constants of superposition

To write down the other components of the EM fields, we use the same superposition coefficients $a$ & $b$. Thus,

$$\tilde{B}_x^{(+)} = \frac{ik}{(\omega^2\epsilon\mu - k^2)}\left(\frac{\partial}{\partial x} B_z(x,y)\right) e^{ikz - i\omega t}$$

and $$\tilde{B}_x^{(-)} = \frac{-ik}{(\omega^2\epsilon\mu - k^2)} \frac{\partial}{\partial x} B_z(x,y)\, e^{-ikz - i\omega t}$$ ($k \to -k$ in the previous expression)

$$\Rightarrow \tilde{B}_x = a\tilde{B}_x^{(+)} + b\tilde{B}_x^{(-)}$$

i.e., $$\boxed{\tilde{B}_x = \frac{ik}{(\omega^2\epsilon\mu - k^2)} \frac{\partial}{\partial x} B_z(x,y) \left( \underset{\sim}{a} e^{ikz} - \underset{\sim}{b} e^{-ikz} \right) e^{-i\omega t}}$$

Similarly, $$\boxed{\tilde{B}_y = \frac{ik}{(\omega^2\epsilon\mu - k^2)} \frac{\partial}{\partial y} B_z(x,y) \left( \underset{\sim}{a} e^{ikz} - \underset{\sim}{b} e^{-ikz} \right) e^{-i\omega t}}$$

Now $$\tilde{E}_y^{(+)} = -\frac{i\omega}{(\omega^2\epsilon\mu - k^2)} \left( \frac{\partial}{\partial x} B_z(x,y) \right) e^{ikz - i\omega t} = -\frac{\omega}{k} \tilde{B}_x^{(+)}$$

and $$\tilde{E}_y^{(-)} = +\frac{\omega}{k} \tilde{B}_x^{(-)} \qquad (k \to -k)$$

$$\Rightarrow \boxed{\tilde{E}_y = \underset{\sim}{a} \tilde{E}_y^{(+)} + \underset{\sim}{b} \tilde{E}_y^{(-)} = \frac{-i\omega \frac{\partial}{\partial x} B_z(x,y)}{(\omega^2\epsilon\mu - k^2)} \left( \underset{\sim}{a} e^{ikz} + \underset{\sim}{b} e^{-ikz} \right) \times e^{-i\omega t}}$$

And, similarly

$$\boxed{\tilde{E}_x = \frac{i\omega}{(\omega^2\epsilon\mu - k^2)} \frac{\partial}{\partial y} B_z(x,y) \left[ \underset{\sim}{a} e^{ikz} + \underset{\sim}{b} e^{-ikz} \right] e^{-i\omega t}}$$

(We have in all our expressions allowed for a general $\epsilon, \mu$ for the medium fillig the waveguide, but it's ok if you replaced them by $\epsilon_0, \mu_0$ as in the text for the purposes of this problem

(c) For a TM mode,

$$\tilde{E}_z = E_z(x,y) \left( \underset{\sim}{a} e^{ikz} + \underset{\sim}{b} e^{-ikz} \right) e^{-i\omega t},$$

$\underset{\sim}{a}, \underset{\sim}{b}$ : again two arbitrary superposition coefficients

Here, by the same approach,

$$\begin{Bmatrix} \tilde{E}_x \\ \tilde{E}_y \end{Bmatrix} = \frac{ik}{(\omega^2\epsilon\mu - k^2)} \begin{Bmatrix} \frac{\partial}{\partial x} E_z(x,y) \\ \frac{\partial}{\partial y} E_z(x,y) \end{Bmatrix} \cdot \left(\underset{\sim}{a} e^{ikz} - \underset{\sim}{b} e^{-ikz}\right) e^{-i\omega t}$$

$$\begin{Bmatrix} \tilde{B}_x \\ \tilde{B}_y \end{Bmatrix} = \frac{-i\omega\,\epsilon\mu}{(\omega^2\epsilon\mu - k^2)} \begin{Bmatrix} (+)\frac{\partial}{\partial y} E_z(x,y) \\ (-)\frac{\partial}{\partial x} E_z(x,y) \end{Bmatrix} \cdot \left(\underset{\sim}{a} e^{ikz} + \underset{\sim}{b} e^{-ikz}\right) \times e^{-i\omega t}$$

overall change in sign, as for the TE mode

(this sign is the same as for $\tilde{E}_z$)

Of course,

$\boxed{\tilde{B}_z = 0}$ for a TM mode

2)

(a) <u>TE mode of the resonant cavity</u>

From the expressions derived in the previous problem, and the requirement that $\tilde{B}_z$, which is the normal component of $\tilde{B}$ on the end plates, must vanish there, we have

$$\tilde{B}_z\Big|_{z=0} = \tilde{B}_z\Big|_{z=d} = 0$$

$$\Rightarrow \underset{\sim}{a} + \underset{\sim}{b} = 0, \quad \underset{\sim}{a} e^{ikd} + \underset{\sim}{b} e^{-ikd} = 0$$

$$\Rightarrow \boxed{\underset{\sim}{b} = -\underset{\sim}{a}}, \text{ so } \underset{\sim}{a}\left(e^{ikd} - e^{-ikd}\right) = 0$$

i.e., $2\underset{\sim}{a} \sin kd = 0 \Rightarrow \sin kd = 0$, since $\underset{\sim}{a} \neq 0$

i.e., $k_p = \frac{p\pi}{d}$, $p = 1, 2, \ldots$

Thus, $\tilde{B}_z = 2i\underset{\sim}{a} B_z(x,y) \sin\left(\frac{p\pi z}{d}\right) e^{-i\omega_{mnp}t}$, where $\omega_{mnp}^2 \epsilon\mu \left(= k_p^2 + \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)$

i.e., $\omega_{mnp} = \frac{1}{\sqrt{\epsilon\mu}} \sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} + \frac{p^2\pi^2}{d^2}}$, $m, n = 0, 1, 2, \cdots$ (but $m = n = 0$ not allowed),

$p = 1, 2, \cdots$ (0 not allowed)

(b) TM mode

Here, we require that $\vec{E}_{tang} = 0$ at $z = 0$ and $z = d$

i.e. $\tilde{E}_x$ (and $\tilde{E}_y$) $= 0$ at $z = 0$, $z = d$

$\Rightarrow \quad \underset{\sim}{a} - \underset{\sim}{b} = 0, \quad \underset{\sim}{a} e^{ikd} - \underset{\sim}{b} e^{-ikd} = 0$

i.e. $\underset{\sim}{a} = \underset{\sim}{b}$ and $2i\underset{\sim}{a} \sin kd = 0$

i.e., $k_p d = p\pi$ (since $a \neq 0$)

or, $k_p = \frac{p\pi}{d}$, $p = 0, 1, 2, \cdots$

(p=0 allowed here)

$$\tilde{E}_z = 2\underset{\sim}{a}\, E_z(x,y) \cos\left(\frac{p\pi}{d} z\right) e^{-i\omega_{mnp} t}, \qquad \omega_{mnp} = \frac{1}{\sqrt{\epsilon\mu}} \sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} + \frac{p^2\pi^2}{d^2}}$$

(arbitrary constant) $m, n = 1, 2, \cdots$ (m or n = 0 not allowed)

$p = 0, 1, 2, \cdots$ (p=0 is allowed)

The other field components are

$$\left\{\begin{matrix} \tilde{E}_x \\ \tilde{E}_y \end{matrix}\right. = \frac{-2\underset{\sim}{a}\, k}{(\omega_{mnp}^2 \epsilon\mu - k_p^2)} \left\{\begin{matrix} \frac{\partial}{\partial x} E_z(x,y) \\ \frac{\partial}{\partial y} E_z(x,y) \end{matrix}\right. \cdot \sin(k_p z)\, e^{-i\omega_{mnp} t}$$

$$\left\{\begin{matrix} \tilde{B}_x \\ \tilde{B}_y \end{matrix}\right. = \mp \frac{2i\underset{\sim}{a}\, \omega_{mnp} \epsilon\mu}{(\omega_{mnp}^2 \epsilon\mu - k_p^2)} \left\{\begin{matrix} \frac{\partial}{\partial y} E_z(x,y) \\ \frac{\partial}{\partial x} E_z(x,y) \end{matrix}\right. \cdot \cos(k_p z)\, e^{-i\omega_{mnp} t}$$

$(\tilde{B}_z = 0)$

Addendum to Part (a)

(Other field components for the $TE_{mnp}$ mode are similarly written down:

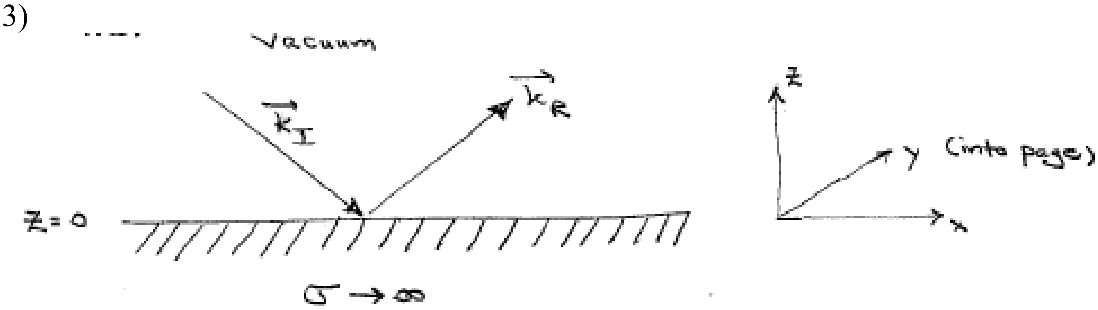
$$\begin{Bmatrix} \tilde{B}_x \\ \tilde{B}_y \end{Bmatrix} = \frac{2ik_p \underset{\sim}{a}}{(\omega_{mnp}^2 \epsilon\mu - k_p^2)} \begin{Bmatrix} \frac{\partial}{\partial x} B_z(x,y) \\ \frac{\partial}{\partial y} B_z(x,y) \end{Bmatrix} \cos(k_p z)\, e^{-i\omega t}$$

and

$$\begin{Bmatrix} \tilde{E}_x \\ \tilde{E}_y \end{Bmatrix} = \mp \frac{2\omega_{mnp} \underset{\sim}{a}}{(\omega_{mnp}^2 \epsilon\mu - k_p^2)} \begin{Bmatrix} \frac{\partial}{\partial y} B_z(x,y) \\ \frac{\partial}{\partial x} B_z(x,y) \end{Bmatrix} \sin(k_p z)\, e^{-i\omega t}$$

$$\left(\tilde{E}_z = 0\right) \quad )$$

3)




$$\vec{E}_I = \hat{e}_y E_0 e^{i(\vec{k}_I \cdot \vec{r} - \omega t)} , \quad \vec{k}_I = k_x \hat{e}_x - k_z \hat{e}_z$$

$$\vec{E}_R = \hat{e}_y E_1 e^{i(\vec{k}_R \cdot \vec{r} - \omega t)} , \quad \vec{k}_R = k_x \hat{e}_x + k_z \hat{e}_z$$

$$\omega = c k_I = c k_R = c\sqrt{k_x^2 + k_z^2}$$

(a) Total electric field

$$\vec{E} = \vec{E}_I + \vec{E}_R = \hat{e}_y \Big( E_0 e^{i(k_x x - k_z z - \omega t)} + E_1 e^{i(k_x x + k_z z - \omega t)} \Big)$$

B.C.: tangential $\vec{E}$ continuous at $z=0$. Since $\vec{E}=0$ inside conductor, we have

normal component of $\vec{E}$ is 0 and thus continuous across boundary, thus implying that there is no surface charge on boundary

$$0 = \vec{E}\Big|_{z=0} = \hat{e}_y \left( E_0 + E_1 \right) e^{i(k_x x - \omega t)}$$

$$\Rightarrow \boxed{E_1 = -E_0}$$

The waves must interfere destructively at $z=0$

$$\therefore \vec{E} = \hat{e}_y E_0 \underbrace{\left( e^{-ik_z z} - e^{ik_z z} \right)}_{-2i \sin k_z z} e^{i(k_x x - \omega t)}$$

$$\boxed{\begin{aligned}&\vec{E} = -i\hat{e}_y 2E_0 \sin k_z z\, e^{i(k_x x - \omega t)} \longleftarrow \text{complex form of total } \vec{E}\\ &\text{Take real part:}\\ &\vec{E} = \hat{e}_y\, 2E_0 \sin k_z z\, \sin(k_x x - \omega t)\end{aligned}}$$

(b) Total magnetic field

<u>Method 1:</u> Use Faraday's law

$$+\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} = +\frac{\partial E_y}{\partial z}\hat{e}_x - \frac{\partial E_y}{\partial x}\hat{e}_z$$

↑ $\vec{E}$ has only a y component

$$= +2E_0 k_z \cos k_z z \sin(k_x x - \omega t)\,\hat{e}_x$$
$$- 2E_0 k_x \sin k_z z \cos(k_x x - \omega t)\,\hat{e}_z$$

Integrate wrt t: We could add a function of position to following, but that wouldn't be a wave

$$\boxed{\begin{aligned}\vec{B} = &\ 2E_0 \frac{k_z}{\omega} \cos k_z z \cos(k_x x - \omega t)\,\hat{e}_x\\ &+ 2E_0 \frac{k_x}{\omega} \sin k_z z \sin(k_x x - \omega t)\,\hat{e}_z\end{aligned}}$$

<u>Method 2:</u> Use magnetic fields for incident and reflected waves.

$$\vec{B}_I = \frac{1}{c}\hat{k}_I \times \vec{E}_I = \frac{1}{\omega}(k_x \hat{e}_x - k_z \hat{e}_z) \times \hat{e}_y E_0 e^{i(\vec{k}_I \cdot \vec{r} - \omega t)}$$

$$= \Big(\frac{k_x}{\omega}\underbrace{\hat{e}_x \times \hat{e}_y}_{\hat{e}_z} - \frac{k_z}{\omega}\underbrace{\hat{e}_z \times \hat{e}_y}_{-\hat{e}_x}\Big) E_0 e^{i(\vec{k}_I \cdot \vec{r} - \omega t)}$$

$$\frac{\vec{k}_I}{c k_I} = \frac{\vec{k}_I}{\omega}$$

$$\vec{B}_I = \left(\frac{k_x}{\omega}\hat{e}_z + \frac{k_z}{\omega}\hat{e}_x\right) E_0\, e^{-ik_z z}\, e^{i(k_x x - \omega t)}$$

$$\vec{B}_R = \frac{1}{\omega}\vec{k}_R \times \vec{E}_R = \frac{1}{\omega}(k_x\hat{e}_x + k_z\hat{e}_z)\times\hat{e}_y\, \overset{-E_0}{\overset{\parallel}{E_1}}\, e^{i(\vec{k}_R\cdot\vec{r} - \omega t)}$$

$$= -\Big(\frac{k_x}{\omega}\underbrace{\hat{e}_x\times\hat{e}_y}_{\hat{e}_z} + \frac{k_z}{\omega}\underbrace{\hat{e}_z\times\hat{e}_y}_{-\hat{e}_x}\Big) E_0\, e^{i(\vec{k}_R\cdot\vec{r} - \omega t)}$$

$$\vec{B}_R = \left(-\frac{k_x}{\omega}\hat{e}_z + \frac{k_z}{\omega}\hat{e}_x\right) E_0\, e^{ik_z z}\, e^{i(k_x x - \omega t)}$$

Total magnetic field:

$$\vec{B} = \vec{B}_I + \vec{B}_R = \frac{k_x}{\omega}\hat{e}_z E_0 \overbrace{\left(e^{-ik_z z} - e^{ik_z z}\right)}^{-2i\sin k_z z} e^{i(k_x x - \omega t)}$$

$$+ \frac{k_z}{\omega}\hat{e}_x E_0 \underbrace{\left(e^{-ik_z z} + e^{ik_z z}\right)}_{2\cos k_z z} e^{i(k_x x - \omega t)}$$

$$\boxed{\vec{B} = \left(-i\,\hat{e}_z\frac{k_x}{\omega}2E_0\sin k_z z + \hat{e}_x\frac{k_z}{\omega}2E_0\cos k_z z\right)e^{i(k_x x - \omega t)}}$$

complex form of total $\vec{B}$

Take real part:

$$\vec{B} = \hat{e}_z\frac{k_x}{\omega}2E_0\sin k_z z\,\sin(k_x x - \omega t)$$

$$+ \hat{e}_x\frac{k_z}{\omega}2E_0\cos k_z z\,\cos(k_x x - \omega t)$$

BC's on $\vec{B}$:

(1) Normal component of $\vec{B}$ continuous at $z=0$. Since $\vec{B}=0$ inside conductor, we have

$$\boxed{0 = \hat{e}_z \cdot \vec{B}\Big|_{z=0} = B_z\Big|_{z=0} \quad \checkmark \quad \text{Works out because } \sin k_z z = 0 \text{ at } z=0}$$

(2) Tangential component of $\vec{B}$ satisfies

$$\underbrace{\vec{B}\cdot\hat{t}\Big|_{z=0^+}}_{\text{just above}} - \underbrace{\vec{B}\cdot\hat{t}\Big|_{z=0^-}}_{\substack{\text{just below} \\ =0}} = \mu_0 \vec{K}\cdot \underbrace{\hat{n}}_{=\hat{e}_z}\times\hat{t}$$

$$\hat{t}=\hat{e}_y: \quad 0 = B_y\Big|_{z=0^+} = \mu_0 \vec{K}\cdot \overbrace{\hat{e}_z\times\hat{e}_y}^{-\hat{e}_x} = -\mu_0 K_x$$

$$\hat{t}=\hat{e}_x: \quad \frac{k_z}{\omega} 2E_0 \cos(k_x x-\omega t) = B_x\Big|_{z=0^+} = \mu_0 \vec{K}\cdot \underbrace{\hat{e}_z\times\hat{e}_x}_{\hat{e}_y} = \mu_0 K_y$$

$$\boxed{\vec{K} = \hat{e}_y \frac{k_z}{\mu_0\omega} 2E_0\cos(k_x x-\omega t)}$$

(c) $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

<u>Method 1:</u>
$$\vec{S} = \frac{1}{\mu_0}\left(\hat{e}_y 2E_0 \sin k_z z \sin(k_x x-\omega t)\right) \times \left(\hat{e}_z \frac{k_x}{\omega} 2E_0 \sin k_z z \sin(k_x x-\omega t) + \hat{e}_x \frac{k_z}{\omega} 2E_0 \cos k_z z \cos(k_x x-\omega t)\right)$$

$$\vec{S} = \frac{k_x}{\mu_0\omega} 4E_0^2 \sin^2 k_z z \sin^2(k_x x-\omega t)\,\underbrace{\hat{e}_y\times\hat{e}_z}_{\hat{e}_x} + \frac{k_z}{\mu_0\omega} 4E_0^2 \sin k_z z \cos k_z z \sin(k_x x-\omega t)\cos(k_x x-\omega t)\,\overbrace{\hat{e}_y\times\hat{e}_x}^{-\hat{e}_z}$$

Now take the time average. Use

$$\langle \sin^2(k_x x - \omega t)\rangle = \frac{1}{2}$$

$$\langle \sin(k_x x - \omega t)\cos(k_x x - \omega t)\rangle = \frac{1}{2}\langle \sin(2k_x x - 2\omega t)\rangle = 0$$

$$\boxed{\langle \vec{S}\rangle = \frac{k_x}{\mu_0 \omega} 2E_0^2 \sin^2(k_z z)\,\hat{e}_x}$$

$$\frac{1}{\mu_0 v_p}$$

Energy propagates in $+x$ direction

Method 2: Use phasor representation

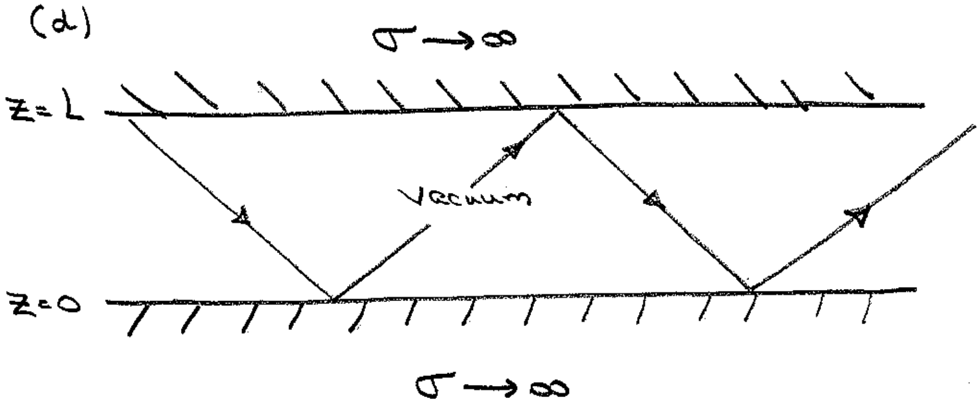
$$\langle \vec{S}\rangle = \frac{1}{2\mu_0}\mathrm{Re}\big(\underbrace{\tilde{\vec{E}}\times\tilde{\vec{B}}^*}\big)$$

$$= \left(-i\hat{e}_y 2E_0 \sin k_z z\right)\times\left(+i\hat{e}_z \frac{k_x}{\omega}2E_0 \sin k_z z + \hat{e}_x \frac{k_z}{\omega}2E_0 \cos k_z z\right)$$

$$= \underbrace{\hat{e}_x \frac{k_x}{\omega}4E_0^2 \sin^2 k_z z}_{\text{real}} + \underbrace{i\hat{e}_z \frac{k_z}{\omega}4E_0^2 \sin k_z z \cos k_z z}_{\text{pure imaginary}}$$

$$\boxed{\langle \vec{S}\rangle = \frac{k_x}{\mu_0 \omega} 2E_0^2 \sin^2 k_z z\,\hat{e}_x}$$

(d)




The boundary conditions say that the tangential electric field must vanish at $z = L$.

$$\Rightarrow \quad 0 = \vec{E}\Big|_{z=L} = \underbrace{\hat{e}_y}_{\text{tangential component}} 2E_0 \sin k_z L \sin(k_x x - \omega t)$$

$$\Rightarrow \quad \sin k_z L = 0 \quad \Rightarrow \quad \boxed{k_z L = n\pi, \quad n = 1, 2, 3, \ldots}$$

Negative integers just change the sign of $\vec{E}$, so they're no different from positive integers; $n=0$ gives no field, so it doesn't count.

(e) Dispersion relation

$$\omega = ck = c\sqrt{k_x^2 + k_z^2} = \boxed{c\left(k_x^2 + (n\pi/L)^2\right)^{1/2} = \omega}$$

Smallest frequency we can have is for $k_x = 0$ and $n = 1$, which gives $\omega = c\pi/L \equiv \omega_{cutoff}$. For $\omega < \omega_{cutoff}$, TE modes cannot propagate down the guide.

Phase velocity:

$$V_p = \frac{\omega}{k_x} = c\left(1+(n\pi/L)^2\right)^{1/2} > c$$

always

Group velocity:

$$V_g = \frac{d\omega}{dk_x} = c\,\frac{1}{\cancel{2}}\,\frac{\cancel{2}k_x}{\left(k_x^2+(n\pi/L)^2\right)^{1/2}} = \frac{c}{\left(1+(n\pi/L)^2\right)^{1/2}} < c$$

always

4) EXTRA CREDIT

Gauge Transformations

We are given a potential pair $(V, \vec{A})$

The electric and magnetic fields associated with these are

$$\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

If we define a new set of potentials via the gauge transformation

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi, \quad V' = V - \frac{\partial \chi}{\partial t} \quad \left(\begin{array}{c}\chi \text{ arbitrary} \\ \text{scalar field}\end{array}\right)$$

Then $\vec{E}$ and $\vec{B}$ are unchanged.

(a) Suppose $\vec{\nabla}\cdot\vec{A} \neq 0$. We want to make a gauge transformation to a new $\vec{A}'$ such that $\vec{\nabla}\cdot\vec{A}' = 0$

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi \Rightarrow \vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\chi = 0$$

Thus we must choose the scalar field which satisfies the differential equation:

$$\nabla^2\chi = -\vec{\nabla}\cdot\vec{A}$$

This is Poisson's equation for $\chi$. $\vec{\nabla}\cdot\vec{A}$ is a given function.

We know $\nabla^2 V = -\rho/\epsilon_0 \Rightarrow V(\vec{r},t) = \frac{1}{4\pi}\int \frac{\rho(\vec{r}',t)/\epsilon_0}{|\vec{r}-\vec{r}'|}d\tau'$

thus $\nabla^2\chi = -\vec{\nabla}\cdot\vec{A} \Rightarrow \boxed{\chi(\vec{r},t) = \frac{1}{4\pi}\int \frac{\vec{\nabla}\cdot\vec{A}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}d\tau'}$

(b) Same idea – we want $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial V'}{\partial t} = 0$

$$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \nabla^2\chi$$

$$\frac{1}{c^2}\frac{\partial V'}{\partial t} = \frac{1}{c^2}\frac{\partial V}{\partial t} - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2}$$

$$\therefore\ \vec{\nabla}\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial V'}{\partial t} = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} + \nabla^2\chi + \frac{1}{c^2}\frac{\partial V}{\partial t} - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = 0$$

Differential equation satisfied by $\chi$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\chi(\vec{r},t) = -\Big(\underbrace{\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}}_{\text{known function}}\Big)$$

We know the solution to equations of this form:

Example: $\left[\begin{array}{l}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)V(\vec{r},t) = -\rho/\epsilon_0 \\ \Rightarrow V(\vec{r},t) = \frac{1}{4\pi}\int \frac{\rho(\vec{r}\,',t_{ret})/\epsilon_0}{|\vec{r}-\vec{r}\,'|}d\tau'\end{array}\right]$

Thus the scalar field which we need is

$$\boxed{\begin{array}{c}\chi(\vec{r},t) = \frac{1}{4\pi}\int \frac{d\tau'}{|\vec{r}-\vec{r}\,'|}\left(\vec{\nabla}\cdot\vec{A}(\vec{r}\,',t_{ret}) + \frac{1}{c^2}\frac{\partial V}{\partial t}(\vec{r}\,',t_{ret})\right) \\ t_{ret} = t - \frac{|\vec{r}-\vec{r}\,'|}{c}\end{array}}$$