Homework #8 *due date is Wednesday 11/5/2013* ***in my mailbox by 5pm***
(see syllabus about late HWs)
(Note that all Griffiths Problems are from 4th Ed.)

1. Consider the general problem of bidirectional propagation inside a cylindrical waveguide. Let the solution for the xy dependence of the B_z field, *i.e.*, $B_z(x, y)$, be known for a forward propagating TE mode of angular frequency ω and propagation constant k (along z).

(a). Show that the same xy dependence must hold for the backward propagating mode with the same magnitude of the propagation constant and frequency.

(b). Use this result to write down the most general solution for B_z for the the TE mode that includes both forward and backward propagating waves. (Two independent superposition constants must be introduced to achieve this.) Derive all other components of the total E and B fields from the so-obtained B_z field. (*Hint:* You must perform the $k \to -k$ substitution in order to derive each of the backward propagating field components from their forward propagating counterparts. Then superpose to obtain each component of the total E or B field.)

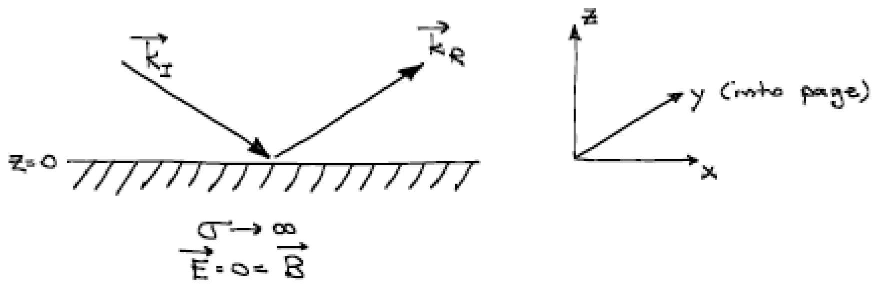
(c). Repeat the calculations of part (b) for a TM mode.

2. Use the general approach developed in problem 1 above to solve 9.40 in Griffiths. Note that the 2 additional conducting boundary conditions must be imposed at the end plates.

3. A waveguide : Consider a monochromatic plane wave incident at an oblique angle on the planar surface ($z = 0$) of a *perfect* conductor (conductivity $\sigma \to \infty$), as shown in the drawing below. The incident wave reflects off the conducting surface to produce a reflected wave. Both the incident wave and the reflected wave are polarized perpendicular to the plane of incidence. The complex forms for the electric fields of the incident and reflected waves are

$$\mathbf{E}_I = \hat{\mathbf{e}}_y E_0 e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)}\,, \qquad \mathbf{k}_I = k_x \hat{\mathbf{e}}_x - k_z \hat{\mathbf{e}}_z\,,$$
$$\mathbf{E}_R = \hat{\mathbf{e}}_y E_1 e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)}\,, \qquad \mathbf{k}_R = k_x \hat{\mathbf{e}}_x + k_z \hat{\mathbf{e}}_z\,,$$

where E_0 and E_1 are *real* amplitudes, and where $\omega = c|\mathbf{k}_I| = c|\mathbf{k}_R| = c\sqrt{k_x^2 + k_z^2}$.



An electromagnetic wave cannot propagate within a perfect conductor; the skin depth is zero, so the wave attenuates before it can get started. Thus we can assume that *in the interior of the perfect conductor, both the electric and magnetic fields are zero, i.e.,* $\mathbf{E} = 0 = \mathbf{B}$.

(a) Using the boundary conditions on the electric field, *show* that the *total* electric field above the conducting plane ($z > 0$) is given by

$$\mathbf{E} = \hat{\mathbf{e}}_y 2E_0 \sin k_z z \sin(k_x x - \omega t) \ .$$

This field describes a standing wave in the z direction (standing because the nodes don't move in the z direction) and a traveling wave in the x direction, with propagates with phase velocity $v_p = k_x/\omega$ (traveling because the nodes move in the x direction).

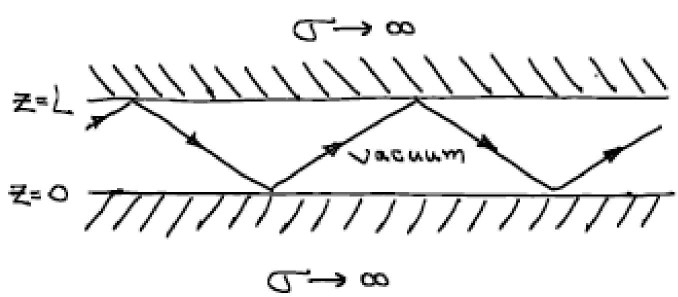
(b) *Show* that the *total* magnetic field $\mathbf{B}$ above the conducting plane is given by

$$\mathbf{B} = \hat{\mathbf{e}}_z \frac{k_x}{\omega} 2E_0 \sin k_z z \sin(k_x x - \omega t) + \hat{\mathbf{e}}_x \frac{k_z}{\omega} 2E_0 \cos k_z z \cos(k_x x - \omega t) \ .$$

Show that the normal component of the total magnetic field satisfies the correct boundary condition and that the boundary condition on the tangential component of the total magnetic field implies that there is a current on the surface of the conductor. *Find* the surface current density $\mathbf{K}$.

(c) *Find* the time-averaged Poynting vector $\langle \mathbf{S} \rangle$ above the conducting plane.

Now suppose that another perfect conductor is placed above the first, the surface of the second conductor being the plane $z = L$, as shown in the drawing below. The two conductors make a "wave guide," which channels the wave. Other examples of wave guides are coaxial cables and optical fibers.



(d) Using the boundary conditions on the electric field, *show* that k_z can only take on a discrete set of values, one value for each positive integer n. *Give* the allowed values of k_z.

Each positive integer n gives electric and magnetic fields that are called a "normal mode" of the wave guide. For this case, where **E**, but not **B**, is perpendicular to the direction of propagation, the normal modes are called "Transverse Electric (TE) modes."

(e) *Give* the dispersion relation—i.e., ω as a function of k_x—for the nth normal mode. *Show* that if $\omega < c\pi/L$, the wave cannot propagate down the wave guide; the frequency $c\pi/L \equiv \omega_{\text{cutoff}}$ is called the *cutoff frequency*. *Find* the phase velocity $v_p = \omega/k_x$ and the group velocity $v_g = d\omega/dk_x$. Does either exceed the speed of light?

4. EXTRA-CREDIT: A problem on gauge transformations. Let V(**r**,t) and $\mathbf{A}(\mathbf{r},t)$ be an arbitrary pair of scalar and vector potentials. The gauge transformation,

$$V' = V - \frac{\partial \chi}{\partial t} ,$$
$$\mathbf{A}' = \mathbf{A} + \nabla \chi ,$$

changes the potentials, but leaves the fields unchanged.

(a) *Coulomb gauge: Show* that we can transform to Coulomb gauge, i.e., make

$$\nabla \cdot \mathbf{A}' = 0 ,$$

by choosing

$$\chi(\mathbf{r},t) = \frac{1}{4\pi} \int d\tau' \, \frac{\nabla \cdot \mathbf{A}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} .$$

(b) *Lorentz gauge: Show* that we can transform to Lorentz gauge, i.e., make

$$\nabla \cdot \mathbf{A}' = -\mu_0 \epsilon_0 \frac{\partial V'}{\partial t} = -\frac{1}{c^2} \frac{\partial V'}{\partial t} ,$$

by choosing

$$\chi(\mathbf{r},t) = \frac{1}{4\pi} \int \frac{d\tau'}{|\mathbf{r}-\mathbf{r}'|} \left(\nabla \cdot \mathbf{A}(\mathbf{r}',t_{\text{ret}}) + \frac{1}{c^2} \left.\frac{\partial V}{\partial t}\right|_{(\mathbf{r}',t_{\text{ret}})} \right) ,$$

where

$$t_{\text{ret}} = t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

is the retarded time.

(Hint: In both parts, find the differential equation satisfied by the gauge field χ, and then solve this equation.)