HW#7 solutions

30% of this HW is extra credit, but try some parts of each problem!

1.

Plane-polarized EM wave incident normally on flat surface of non-permiable medium of conductivity σ + permittivity ϵ. Show that for a good conductor, the power reflection coeff is approximately

$$R \approx 1 - \frac{2\omega}{c}\delta \; ; \quad \text{where } \delta \equiv \text{skin depth}$$

Normal incidence:

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \quad ; \mu_1 = \mu_2 = \mu_0$$

We take $n_1 = 1$ and $\tilde{n}_2 = \frac{\tilde{k}}{(\omega/c)} = \frac{k(1+i)}{(\omega/c)}$

$$\tilde{n}_2 = \frac{\sqrt{\frac{\mu_0 \sigma \omega}{2}}}{\omega/c}(1+i)$$

$$= c\sqrt{\frac{\mu_0 \sigma}{2\omega}}\,(1+i)$$

For a good conductor σ is large and $\tilde{n}_2$ is large. So expand R in powers of n_1/n_2:

$$R = \left|\left(\frac{n_1 - n_2}{n_1 + n_2}\right)\right|^2 \approx \left|\frac{1 - n_1/n_2}{1 + n_1/n_2}\right|^2$$

$$\approx \left| \frac{1 - 2n_1/n_2 + \cdots}{1 + 2n_1/n_2 + \cdots} \right|^2$$

Take $x = 2n_1/n_2$, $f(x) = \dfrac{1-x}{1+x}$

Taylor expand about $x = 0$:

$$f(x) \approx f(0) + f'(x)\Big|_{x=0} x + \cdots$$

$$\approx 1 + \left[-\frac{1}{1+x} - \frac{(1-x)}{(1+x)^2} + \cdots \right]\Bigg|_{x=0} x$$

$$\approx 1 + \left(\frac{-1-x-1+x}{(1+x)^2} \right)\Bigg|_{x=0} x \approx 1 - 2x$$

$$\Rightarrow \left| \frac{1 - 2n_1/n_2}{1 + 2n_1/n_2} \right| \approx \left| 1 - 4\frac{n_1}{n_2} \right|$$

$$\approx \left| 1 - \frac{4}{\frac{1}{\delta}\frac{c}{\omega}(1+i)} \right|$$

, where we've used the fact that the skin depth in a <u>good</u> conductor is $\delta = \dfrac{\lambda_2}{2\pi} = \dfrac{1}{k_2}$

$$k_2 = \sqrt{\frac{\mu_0 \sigma \omega}{2}}$$

$$\Rightarrow R \approx \left| 1 - 4\delta\frac{\omega}{c}\frac{(1-i)}{2} \right| =$$

$$A + iB$$

$$\approx \left| 1 - 2\frac{\delta\omega}{c} + 2i\frac{\delta\omega}{c} + \dots \right|$$

$$\approx \sqrt{\left(1 - 2\frac{\delta\omega}{c}\right)^2 + \left(2\frac{\delta\omega}{c}\right)^2}$$

$$\approx \sqrt{1 - 4\frac{\delta\omega}{c} + 16\left(\frac{\delta\omega}{c}\right)^2} \approx (1+x)^{1/2}$$

$$\approx 1 + \frac{1}{2}x \approx 1 + \frac{1}{2}\left(-4\frac{\delta\omega}{c} + 16\left(\frac{\delta\omega}{c}\right)^2\right)$$

$$\boxed{R \approx 1 - 2\frac{\delta\omega}{c} + \mathcal{O}\left(\frac{\delta\omega}{c}\right)^2}$$

2. see # 4, they were identical

3. Silver as a conductor:

Silver as a conductor

D.C. conductivity $\sigma_0 = 6.17\times10^{+7} /\Omega m$

Ag: Density (at STP) ~ $10.5\ g/cm^3 \cong 10^4\ kg/m^3 = \rho_M$

Valence electrons : 1

number density $= N_e = \rho_M \frac{1}{\text{mass Ag}}$

Atomic weight Ag: 108 amu, 1 amu = mass of proton $= 1.7\times10^{-27}$ kg

$\Rightarrow m_{Ag} = 1.83\times10^{-25}$ kg

$\therefore N = \left(10^4 \frac{kg}{m^3}\right)\left(\frac{1}{1.83\times10^{-25} kg}\right) = 5.5\times10^{28}\ m^{-3}$

The D.C. conductivity $\sigma_0 = \frac{f Ne^2}{m_e \gamma}$

f = # of free electrons/atom = 1

$\Rightarrow$ Collision rate $\gamma = \frac{Ne^2}{m_e \sigma_0} = \frac{(5.5\times10^{28} m^{-3})(1.6\times10^{-19}\ Coul)^2}{(6.17 \frac{10^{+7}}{ohm\ m})(9\times10^{-31}\ kg)}$

$\Rightarrow \boxed{\gamma = 2.5\times10^{13}\ s^{-1}}$ $\quad \frac{\gamma}{2\pi} = 4\times10^{12}$ Hz (infra-red)

Plasma frequency: $\omega_p \equiv \sqrt{\frac{Ne^2}{\epsilon_0 m_e}} = \sqrt{\frac{\gamma \sigma_0}{\epsilon_0}}$

$\Rightarrow \omega_p = \sqrt{\frac{(2.5\times10^{13}\ s^{-1})(6.17\times10^7 \frac{Coul}{s\ Volt\ m})}{(8.85\times10^{-12} \frac{Coul}{m\ Volt})}}$

$\Rightarrow \boxed{\omega_p = 1.3\times10^{16} s^{-1}}$ $\quad \frac{\omega_p}{2\pi} = 2.1\times10^{15}$ Hz

(Ultraviolet)

(b) In the Drude model, the conductivity is

$$\tilde{\sigma}(\omega) = \frac{f\,Ne^2/m}{\gamma - i\omega} = \frac{\sigma_0}{1 - i\frac{\omega}{\gamma}}$$

- For frequencies small compared to the collision rate $\frac{\omega}{\gamma} << 1$ we essentially have the D.C. conductivity
  $\tilde{\sigma}(\omega) \approx \sigma_0$
- For frequencies large compared to the collision rate $\frac{\omega}{\gamma} >> 1$ the conductivity become imaginary and the metal looks like a plasma
  $$\tilde{\sigma}(\omega) \approx i\,\frac{\sigma_0 \gamma}{\omega} = i\,\frac{\omega_p^2}{\omega}\epsilon_0$$

The general complex index of refraction is

$$\tilde{n}(\omega) = \frac{c}{\omega}\tilde{k} = \frac{c}{\omega}\sqrt{\mu_0\epsilon_0\omega^2 + i\omega\tilde{\sigma}\mu_0} \quad \left(\begin{array}{c}\text{Taking}\\ \epsilon = \epsilon_0,\ \mu = \mu_0\end{array}\right)$$

$$\Rightarrow \tilde{n}(\omega) = \sqrt{1 + i\,\frac{\tilde{\sigma}(\omega)}{\omega\epsilon_0}}$$

- Typical microwave frequency: $5\times10^{10}\,Hz = \nu$
  $$\omega = 3.1\times10^{11}\,s^{-1} << \gamma$$
  $\Rightarrow \tilde{\sigma}(\omega) \approx \sigma_0$. Furthermore $\frac{\sigma_0}{\omega\epsilon_0} >> 1$ (Excellent conductor)
  $$\therefore \boxed{\tilde{n}(\omega) \approx \sqrt{i\,\frac{\sigma_0}{\omega\epsilon_0}} \approx \sqrt{\frac{\sigma_0}{2\omega\epsilon_0}}\,(1+i) = 4.7\times10^3\,(1+i)}$$

- Typical visible frequency: $\nu = 5\times10^{14}\,Hz$
  $$\omega = 3.1\times10^{15}\,s^{-1} >> \gamma$$
  $\Rightarrow \tilde{\sigma}(\omega) \approx i\,\frac{\omega_p^2}{\omega}\epsilon_0 \Rightarrow \tilde{n}(\omega) \cong \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$ Plasma dispersion

Since $\omega < \omega_p$, below cutoff $\Rightarrow \boxed{\tilde{n}(\omega) = i\sqrt{\frac{\omega_p^2}{\omega^2} - 1} = 4i}$

- Typical X-ray frequency $\nu = 5\times10^{18}$ Hz

$$\Rightarrow \omega = 3.1\times10^{19}\ s^{-1} \gg \gamma$$

$$\tilde{\sigma}(\omega) \approx i\frac{\omega_p^2}{\omega}\epsilon_0 \quad\Rightarrow\quad \tilde{n}(\omega) \cong \sqrt{1-\frac{\omega_p^2}{\omega^2}} \quad \text{Plasma}$$

Since $\omega_p \ll \omega$ this plasma is excited well above cuttoff

$$\Rightarrow \tilde{n}(\omega) \approx 1 - \frac{1}{2}\left(\frac{\omega_p}{\omega}\right)^2 = 1 - 8.8\times10^{-8}$$

So at X-ray frequencies this simple model says that Silver is essentially transparent. Of course there are lots of other effects we have neglected such as absorption by bound electrons and Bragg-scattering

(c) At a radio frequency $\nu = 1$ MHz

$$\omega = 6.2\times10^{7}\ s^{-1} \ll \gamma \quad\text{and}\quad \frac{\sigma_0}{\omega\epsilon_0} \gg 1 \ \text{(Good Conductor)}$$

$$\Rightarrow \tilde{\sigma}(\omega) \approx \sigma_0 \qquad \tilde{n}(\omega) \approx \sqrt{\frac{\sigma_0}{2\omega\epsilon_0}}\,(1+i)$$

Skin depth (distance field decays to $1/e$)

$$\boxed{d = \frac{1}{k_I} = \frac{1}{\frac{\omega}{c}n_I(\omega)} = \frac{1}{\sqrt{\frac{\omega\mu_0\sigma_0}{2}}} = 6\times10^{-5}\,m = 60\,\mu m}$$

4. Waves in Plasmas:

Waves in Plasmas. Neutral plasma with electron density N_e. At high ν's ions are very heavy & can be considered fixed; e^-'s carry current.

(a) Using Maxwell's Eqns derive the wave eqn for $\vec{E}$:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0 \frac{\partial \vec{J}}{\partial t}$$

For our case of a neutral plasma we have

$\rho = 0$ (no net charge)

$\vec{J} = \rho_e \vec{v}_e = N_e e \vec{v}_e$; current is carried by e^-'s.

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\cancelto{0}{\vec{\nabla}\cdot\vec{E}}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{B}$$

$$-\nabla^2\vec{E} = -\mu_0\frac{\partial \vec{J}}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\rightarrow \boxed{\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{E} = \mu_0\frac{\partial \vec{J}}{\partial t}}$$

(b) Monochromatic plane wave $\vec{\tilde{E}} = \mathrm{Re}\left[\vec{\tilde{E}} e^{-i\omega t}\right]$ and ignoring collisions between e^-'s, show that

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\vec{E}} = \frac{\omega_p^2}{c^2}\tilde{\vec{E}} \; ; \; \omega_p = \sqrt{\frac{N_e e^2}{\epsilon_0 m_e}}$$

(plasma freq.)

We can use the wave eqn but need relationship between $\vec{E}$ + $\vec{J}$. Recall that

$\vec{J} = N_e e \vec{v}_e$ and the eqn of motion for the e^-'s, that are collisionless, is:

$$m_e \frac{d\vec{v}_e}{dt} = e\vec{E}$$ (no restoring force because e^-'s are free)

This yields:

$$\frac{d\vec{v}_e}{dt} = \frac{e}{m_e}\vec{E}$$

or

$$e N_e \frac{d\vec{v}_e}{dt} = \frac{\partial \vec{J}}{\partial t} \text{ (from above)} = \frac{e^2 N_e}{m_e}\vec{E} = \epsilon_0 \omega_p^2 \vec{E}$$

<u>Now</u> we can use wave eqn:

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\vec{E} = \mu_0 \frac{\partial \vec{J}}{\partial t} = \mu_0 \epsilon_0 \omega_p^2 \vec{E} = \frac{\omega_p^2}{c^2}\vec{E}$$

Sub-ing our soln for the monochromatic waves:

$$\boxed{\left(\nabla^2 - \frac{(-i\omega)^2}{c^2}\right)\tilde{\vec{E}} = \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\tilde{\vec{E}} = \frac{\omega_p^2}{c^2}\tilde{\vec{E}}}$$

Ⓒ Derive the dispersion relation:

$$k = \sqrt{\omega^2 - \omega^2}/c \quad + \text{ sketch graph of } \omega(k)$$

Using our boxed eqn, we substitute a plane wave solⁿ:

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\Rightarrow \quad -k^2 + \frac{\omega^2}{c^2} = \frac{\omega_p^2}{c^2} \quad \text{from which we get}$$

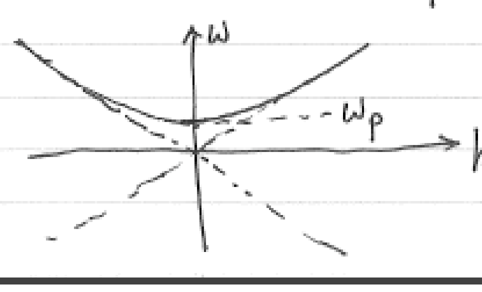
$$\boxed{k = \frac{\sqrt{\omega^2 - \omega_p^2}}{c}}$$

If $\underline{\omega < \omega_p}$, $k = i\beta$ with $\beta = \sqrt{\omega_p^2 - \omega^2}/c$ and we get attenuation:

$$\vec{E} = \mathrm{Re}\left(\tilde{\vec{E}}_0 e^{-\beta z} e^{-i\omega t}\right)$$

$$\vec{E} = \vec{E}_0 e^{-\beta z} \cos(\omega t) \quad \text{(evanescent wave)}$$

Plot of $\omega(k) = \sqrt{c^2 k^2 + \omega_p^2}$



phase vel. $v =$

Here: $\frac{\omega}{k} = c\left(1 + \left(\frac{\omega_p}{k}\right)^2\right)^{1/2} > c$

But $\frac{d\omega}{dk} = \frac{c}{\left(1 + \left(\frac{\omega_p}{\omega}\right)^2\right)^{1/2}} < c$

group vel.

5)

Anomalous Dispersion in Dielectrics.

Group vel in an absorptive ~~dielectr~~ic dispersive medium:

$$v_g = \frac{1}{\frac{dk_R(\omega)}{d\omega}} \; ; \; k_R(\omega) \equiv \text{real part of wave \#}.$$

(a) Show that $v_g(\omega) = \dfrac{c}{n_R(\omega) + \omega\,(dn_R/d\omega)}$

Use:

$$k_R(\omega) = \frac{\omega}{c}\, n_R(\omega) \; ; \; n_R(\omega) \equiv \text{real part of index of refraction}$$

$$\rightarrow \frac{dk_R}{d\omega} = \frac{n_R}{c} + \frac{\omega}{c}\frac{dn_R}{d\omega}$$

$$\rightarrow \boxed{v_g = \frac{c}{n_R(\omega) + \omega\, dn_R/d\omega}}$$

b) For a single resonance in the Lorentz Model we found

$$n_R(\omega) = 1 + \frac{\omega_p^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2},$$

or, near resonance: $n_R(\omega) \approx 1 + A\left(\dfrac{-\Delta}{\Delta^2 + \Gamma^2/4}\right)$

with

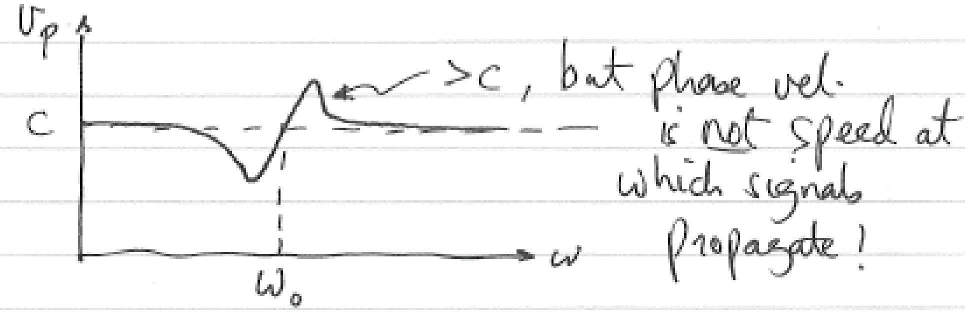
$$\Delta \equiv \omega - \omega_0 \text{ and } A = \frac{\omega_p^2}{2\omega_0}$$

ω_0

Phase velocity:

$$v_p(\omega) = \frac{c}{n_R(\omega)} = \frac{c}{1 + A\left(\frac{-\Delta}{\Delta^2 + \Gamma^2/4}\right)}$$

Sketch:



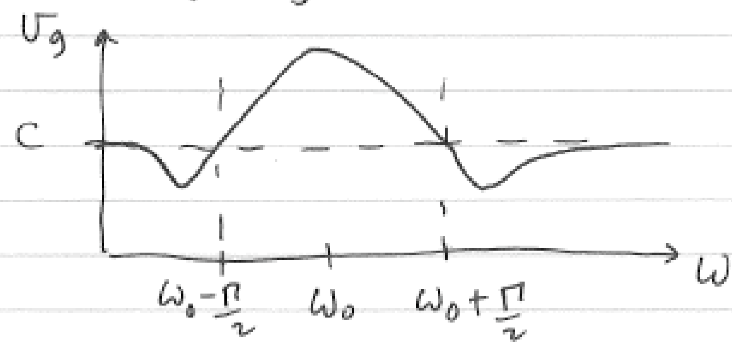
(c) Show that in the region of anomalous dispersion, i.e. when $dn_R/d\omega < 0$, even the group vel $v_g > c$! What is the condition on $n_R(\omega)$?

$$v_g(\omega) = \frac{c}{n_R(\omega) + \omega(dn_R/d\omega)}$$

and near resonance, where anomalous dispersion occurs, we use

$$n_R(\omega) \approx 1 + A\left(\frac{-\Delta}{\Delta^2 + \Gamma^2/4}\right) \quad \text{(part b)}$$

Sketch of $v_g(\omega)$ near $\omega = \omega_0$:



Condition for $v_g > c$ is simply :

$$\boxed{n_R(\omega) + \omega \frac{dn_R}{d\omega} < 1}$$

The region within $\omega_0 - \frac{\Gamma}{2} < \omega < \omega_0 + \frac{\Gamma}{2}$ is within the absorption line. References for further reading on this :

- L. Brillouin, "Wave Propagation + Group Velocity" (Academic Press, NY, 1952)
- R. Y. Chiao et al., Proc. of the Conference, "Fundamental Problems in Quantum Theory" (Ann. NY Acad. Sci 1994)