HW #8 Solutions

1) Birefringence.

a) $\tilde{\vec{E}} = \hat{y} E_0 e^{i(kx-\omega t)}$; $\tilde{\vec{D}} = \hat{y}\,\epsilon_y \tilde{E}_y$

Linearly polarized along $\hat{y}$, propagating in $\hat{x}$ direction.

We use phasors to write down general form for $\tilde{\vec{B}} = \tilde{\vec{B}}_0 e^{i(kx-\omega t)}$

Maxwell's Eqns: $\vec{\nabla}\cdot\tilde{\vec{D}} = 0 \rightarrow ik\hat{x}\cdot\tilde{\vec{D}} = 0$ along $\hat{y}$

Both are satisfied as expected. $\vec{\nabla}\cdot\tilde{\vec{B}} = 0 \rightarrow ik\hat{x}\cdot\tilde{\vec{B}} = 0$

↳ NOT along x

$\vec{\nabla}\times\tilde{\vec{E}} = -\frac{\partial \tilde{B}}{\partial t} \rightarrow ik\hat{x}\times\tilde{\vec{E}} = i\omega \tilde{B}$

which gives $\boxed{\tilde{\vec{B}} = \frac{k}{\omega}\hat{x}\times\tilde{\vec{E}} = \frac{k}{\omega}\hat{z}E_0 e^{i(kx-\omega t)}}$

Good, now use last M.E. to find v_y, phase vel.

$\vec{\nabla}\times\tilde{\vec{B}} = \mu_0 \frac{\partial \tilde{\vec{D}}}{\partial t} = \mu_0 \epsilon_y \frac{\partial \tilde{\vec{E}}_y}{\partial t}$

$$\left.\begin{aligned} ik\hat{x}\times\tilde{\vec{B}} &= -i\omega\mu_0\epsilon_y \tilde{\vec{E}} \\ \underbrace{\tfrac{k}{\omega}\hat{x}\times(\hat{x}\times\tilde{\vec{E}})}_{} &= -\frac{k}{\omega}\tilde{\vec{E}} \end{aligned}\right\} \rightarrow \boxed{\frac{k^2}{\omega^2} = \mu_0\epsilon_y = \frac{1}{v_y^2}}$$

So, $\boxed{\begin{aligned} \tilde{\vec{B}} &= \frac{1}{v_y}\,\hat{z}\,E_0\,e^{i(kx-\omega t)} \\ v_y &= \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_y}} \end{aligned}}$

b) Wave linearly polarized along z:

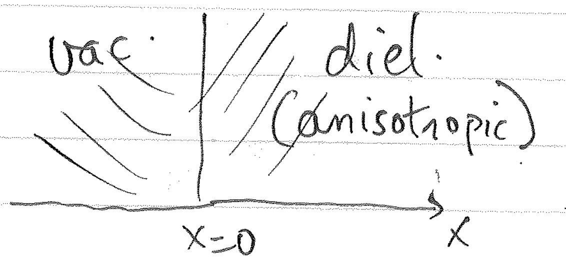
$$\tilde{\vec{E}} = \hat{z}\,E_0\,e^{i(kx-\omega t)}$$

; using part a)

results, we get: $\tilde{\vec{B}} = \frac{k}{\omega}\,\hat{x}\times\tilde{\vec{E}} = -\frac{1}{v_z}\,\hat{y}\,E_0\,e^{i(kx-\omega t)}$

where now $v_z = \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \epsilon_z}}$

c)



Take wave coming from left into dielectric. At interface, wave is linearly polarized along: $\frac{1}{\sqrt{2}}(\hat{y}+\hat{z})$

At $x=0$:

$$\tilde{\vec{E}} = \frac{1}{\sqrt{2}}(\hat{y}+\hat{z})\,E_0\,e^{-i\omega t}$$

We use superposition plus results of parts a) + b) to write down the field in $x>0$ region:

$$\tilde{\vec{E}} = \frac{1}{\sqrt{2}}\,\hat{y}\,E_0\,e^{i(k_y x-\omega t)} + \frac{1}{\sqrt{2}}\,\hat{z}\,E_0\,e^{i(k_z x-\omega t)}$$

Where $k_y = \frac{\omega}{v_y}$ (from part a)

and $k_z = \frac{\omega}{v_z}$ " " b)

Thus: $$\tilde{\vec{E}} = \frac{E_0}{\sqrt{2}}\left(\hat{y}\, e^{i\omega(x/v_y - t)} + \hat{z}\, e^{i\omega(x/v_z - t)}\right)$$

$$\boxed{\tilde{\vec{E}} = \frac{E_0}{\sqrt{2}} e^{-i\omega t}\left(\hat{y}\, e^{i\omega x/v_y} + \hat{z}\, e^{i\omega x/v_z}\right)}$$

Note that although the wave was linearly polarized when it started entering the dielectric, now that it is in the medium its ONLY linearly polarized at points given by:

$$e^{i\omega x/v_y} = e^{i\omega x/v_z}$$

or

$$\boxed{\omega x\left(\frac{1}{v_y} - \frac{1}{v_z}\right) = 2\pi n} \quad \text{any integer } n \; !$$

d) Assume $v_y > v_z$. At $x=0$ it starts linearly polarized along $\frac{1}{\sqrt{2}}(\hat{y} + \hat{z})$

i) Min. x for which it is right circular polarized? From HW #7 we showed that a right circ. polar. gave:

$$\tilde{\vec{E}} = E_0(\hat{y} + i\hat{z}) \quad \text{(For example)}$$

So, we require: $e^{i\omega x/v_z} = i e^{i\omega x/v_y}$

or $e^{i\omega x\left(\frac{1}{v_z} - \frac{1}{v_y}\right)} = i = e^{i\pi/2}$

$$\Rightarrow \omega x\left(\frac{1}{v_z} - \frac{1}{v_y}\right) = \frac{\pi}{2} + 2n\pi \quad \text{where } n = 0, 1, 2 \ldots$$

The min. x where this holds is when $n=0$:

$$\omega x\left(\frac{1}{v_z} - \frac{1}{v_y}\right) = \frac{\pi}{2}$$

or $\boxed{x = \frac{\pi}{2\omega} \cdot \frac{v_y v_z}{v_y - v_z}}$ } A boundary at this position is called a quarter-wave plate

because the medium introduced a phase shift of 90° ($\pi/2$), or $\frac{1}{4}$ of a cycle between the y & z polarizations.

ii) Smallest value of x such that wave ~~has a right-circular polar.~~ is linearly polar. along $\frac{1}{\sqrt{2}}(\hat{y} - \hat{z})$

Now we have $e^{i\omega x/v_z} = -e^{i\omega x/v_y}$ from $Re(e^{i\pi}) = -1$

and $\omega x\left(\frac{1}{v_z} - \frac{1}{v_y}\right) = \pi, 3\pi, 5\pi, \text{etc}$

$$\rightarrow \boxed{x = \frac{\pi}{\omega} \frac{v_y v_z}{v_z - v_y}}$$ is the min. x for which this is true

→ Half-wave plate boundary due to 180° (π), or $\frac{1}{2}$ ~~wave~~ of a cycle phase shift between y + z polar.

iii) Min x for which wave has left circ. polar.

$$\vec{E} = E_0(\hat{y} - i\hat{z})$$

So now $e^{i\omega x\left(\frac{1}{v_z} - \frac{1}{v_y}\right)} = -i = e^{i3\pi/2} = i \sin \frac{3\pi}{2}$

$$\rightarrow \omega x \left(\frac{v_y - v_z}{v_y v_z}\right) = \frac{3\pi}{2}$$ is the condition for min. x that gives ~~satisfies~~ left circ. polar.

$$\boxed{x = \frac{3\pi}{2\omega} \frac{v_y v_z}{v_y - v_z}}$$

2) Refer to problem for forms of $\vec{\tilde{E}}$ & $\vec{\tilde{B}}$ at various boundaries.

a) B.C. at $x=0$ gives 2 conditions between $\tilde{E}_I, \tilde{E}_R, \tilde{C}$ & $\tilde{D}$:

B.C.'s are $\vec{E}_{\parallel}$ and $\vec{H}_{\parallel} = \vec{B}_{\parallel}/\mu_0$ are continuous

$$\rightarrow \left(\vec{\tilde{E}}_I + \vec{\tilde{E}}_R\right)\Big|_{x=0} = \left(\vec{\tilde{E}}_2^{(R)} + \vec{\tilde{E}}_2^{(L)}\right)\Big|_{x=0}$$

$$E_I e^{-i\omega t} + \tilde{E}_R e^{-i\omega t} = \tilde{C} e^{-i\omega t} + \tilde{D} e^{-i\omega t}$$

$$\boxed{E_I + \tilde{E}_R = \tilde{C} + \tilde{D}}$$

Similarly for $H_{\parallel}$ or $\frac{\vec{B}_{\parallel}}{\mu_0}$: $\frac{E_I}{c} - \frac{\tilde{E}_R}{c} = \frac{\tilde{C}}{v} - \frac{\tilde{D}}{v}$

$$\boxed{E_I - \tilde{E}_R = n\left(\tilde{C} - \tilde{D}\right)}, \quad n = c/v$$

b) B.C. at $x=L$ give 2 relns between $\tilde{E}_I, \tilde{E}_R$, $\tilde{E}_T$, $\tilde{C}$ and $\tilde{D}$. Same B.C.'s as in a) except at $x=L$:

$$\left(\vec{\tilde{E}}_2^{(R)} + \vec{\tilde{E}}_2^{(L)}\right)\Big|_{x=L} = \vec{\tilde{E}}_T\Big|_{x=L} \Rightarrow \tilde{C} e^{ik_2 L} + \tilde{D} e^{-ik_2 L} = \tilde{E}_T e^{ikL}$$

$$\boxed{\tilde{C} + \tilde{D}e^{-2ik_2L} = \tilde{E}_T e^{i(k-k_2)L}}$$

For $\vec{B}_{\parallel}$:

$$\frac{1}{v}\tilde{C}e^{ik_2L} - \cancel{4}\frac{1}{v}\tilde{D}e^{-ik_2L} = \frac{\tilde{E}_T}{c}e^{ikL}$$

$$\rightarrow \boxed{n(\tilde{C} - \tilde{D}e^{-2ik_2L}) = \tilde{E}_T e^{i(k-k_2)L}}$$

$n = c/v$

c) Show that when $L = \frac{n}{2}\lambda_2$ ~~lm~~, λ_2 in glass, then all of the wave is transmitted.

$$L = \frac{n\lambda_2}{2} = \frac{n}{2}\cdot\frac{2\pi}{k_2} = \frac{n\pi}{k_2}$$

So, $\boxed{k_2 L = n\pi}$

Using 1st boxed eqn above: $e^{-2ik_2L} = e^{-i2n\pi} = 1$

~~Using 2nd boxed eqn in part a):~~

This gives:

$$(\tilde{C} + \tilde{D}) = \tilde{E}_T e^{i(k-k_2)L}$$

~~Using 2nd boxed eqn in a):~~ $\cancel{E_I + \tilde{E}_R = \tilde{E}_T e^{i(k-k_2)L}}$

~~or~~

$\cancel{n(\tilde{C} - \tilde{D}) = \tilde{E}}$

Using 2nd boxed eqn in b):

$$n(\tilde{C}-\tilde{D}) = \tilde{E}_T e^{-i(k-k_2)L}$$

Using boxed eqns from a) we get:

$$E_I + \tilde{E}_R = \tilde{C}+\tilde{D} = \tilde{E}_T e^{i(k-k_2)L}$$

$$E_I - \tilde{E}_R = n(\tilde{C}-\tilde{D}) = \tilde{E}_T e^{i(k-k_2)L}$$

only way is if $\boxed{\tilde{E}_R = 0}$, giving:

$$\boxed{E_I = \tilde{E}_T e^{i(k-k_2)L}}$$

$\Rightarrow$ No reflected ~~way~~ wave; fully transmitted.

d) Show that when $L = \frac{n}{2}\lambda_2 + \frac{\lambda_2}{4}$ that

$$\tilde{E}_R = \frac{1-n^2}{1+n^2} E_I \ , \quad \tilde{E}_T = \frac{2n}{1+n^2} E_I e^{-i(k-k_2)L}$$

$$L = \left(\frac{n}{2}+\frac{1}{4}\right)\lambda_2 = \left(\frac{n}{2}+\frac{1}{4}\right) \cancel{k_2} \frac{2\pi}{k_2}$$

$$L = \left(n+\frac{1}{2}\right)\frac{\pi}{k_2} \rightarrow k_2 L = \left(n+\frac{1}{2}\right)\pi$$

and $$e^{-i2k_2L} = e^{-2i(n+\frac{1}{2})\pi}$$

$$= \underbrace{e^{-2\pi i n}}_{1}\ \underbrace{e^{-i\pi}}_{\cos\pi = -1} = -1$$

Now our boxed eqns in a) & b) become:

$$E_I + \tilde{E}_R = \tilde{C} + \tilde{D} = \frac{\tilde{E}_T}{n} e^{i(k-k_2)L}$$

$\cancel{E_I}$

2nd box in part b)

$$E_I - \tilde{E}_R = n(\tilde{C} - \tilde{D}) = n\tilde{E}_T e^{i(k-k_2)L}$$

1st box in part b)

Add these:

$$2E_I = \left(\frac{1}{n} + n\right)\tilde{E}_T e^{i(k-k_2)L}$$

$$\rightarrow \boxed{\tilde{E}_T = \frac{2n E_I}{1+n^2} e^{-i(k-k_2)L}}$$

Sub back into one of the eqns above:

$$\tilde{E}_R = -E_I + \frac{2E_I}{1+n^2} = \frac{-E_I - n^2 E_I + 2E_I}{1+n^2}$$

$$\boxed{\tilde{E}_R = \frac{1-n^2}{1+n^2}\cdot E_I}$$

e) In terms of ~~$\mp$~~ E_I there are 4 eqns & 4 unknowns: $\tilde{E}_R$, $\hat{C}$, $\tilde{D}$, $\tilde{E}_T$

Write them as follows:

$$n E_I + n\tilde{E}_R = n\hat{C} + n\tilde{D} \qquad (1^{st} \text{ in a)})$$

$$E_I - \tilde{E}_R = n\hat{C} - n\hat{D} \qquad (2^{nd} \text{ in a)})$$

$$n\tilde{F}_T \equiv n\tilde{E}_T e^{i(k-k_2)L} = n\hat{C} + n\tilde{D} e^{-2ik_2L}$$

$$\tilde{F}_T \equiv \tilde{E}_T e^{i(k-k_2)L} = n\hat{C} - n\hat{D} e^{-2ik_2L}$$

Add 1^{st} 2: $\left\{ \begin{array}{l} (n+1)E_I + (n-1)\tilde{E}_R = 2n\tilde{C} \\ (n+1)\tilde{F}_T = 2n\tilde{C} \end{array} \right.$

" 2^{nd} 2:

$$\Rightarrow (n+1)E_I + (n-1)\tilde{E}_R = (n+1)\tilde{F}_T$$

Subtract 1^{st} 2: $\left\{ \begin{array}{l} (n-1)E_I + (n+1)\tilde{E}_R = 2n\hat{D} \\ (n-1)\tilde{F}_T = 2n\hat{D} e^{-2ik_2L} \end{array} \right.$

" 2^{nd} 2:

$$\Rightarrow (n-1)E_I + (n+1)\tilde{E}_R = (n-1)\tilde{F}_T e^{+2ik_2L}$$

Eliminate $\tilde{F}_T$ from these eqns:

- mult. top by $(n-1)e^{2ik_2L}$, bottom by $(n+1)$ & subtract:

$$\left[(n-1)(n+1)E_I + (n-1)^2 \tilde{E}_R\right] e^{2ik_2L}$$

$$-(n-1)(n+1)E_I - (n+1)^2 \tilde{E}_R = 0$$

$$(n-1)(n+1)\left(e^{2ik_2L} - 1\right)E_I$$

$$+\left[(n-1)^2 e^{2ik_2L} - (n+1)^2\right]\tilde{E}_R = 0$$

$$\boxed{\frac{(n^2-1)\left(e^{2ik_2L}-1\right)}{\left[(n+1)^2-(n-1)^2 e^{2ik_2L}\right]} = \frac{\tilde{E}_R}{E_I}}$$

Sub this back into $(n+1)E_I + (n-1)\tilde{E}_R = (n+1)\tilde{F}_T$

$$(n+1) + (n-1)\frac{\tilde{E}_R}{E_I} = (n+1)\frac{\tilde{F}_T}{E_I}$$

$$1 + \frac{(n-1)}{(n+1)}\frac{(n^2-1)\left(e^{2ik_2L}-1\right)}{\left[(n+1)^2-(n-1)^2 e^{2ik_2L}\right]} = \frac{\tilde{F}_T}{E_I} = \frac{\tilde{E}_T}{E_I}e^{i(k-k_2)L}$$

A mess! The n^2-1 term $= (n+1)(n-1)$ so the $(n+1)$ cancels $(n+1)$ in denominator:

$$= 1 + \frac{(n-1)^2\left(e^{2ik_2L}-1\right)}{\left[(n+1)^2-(n-1)^2 e^{2ik_2L}\right]} = \frac{\tilde{F}_T}{E_I}$$

cancel

$$\frac{(n+1)^2 - (n-1)^2 e^{(\,)} + (n-1)^2\left(e^{(\,)} - 1\right)}{\left[(n+1)^2 - (n-1)^2 e^{2ik_2L}\right]} = \frac{\tilde{F}_T}{E_I}$$

$$= \frac{(n+1)^2 - (n-1)^2}{[\qquad]} = \text{"}$$

$$= \frac{n^2 + 2n + 1 - n^2 + 2n - 1}{[\qquad]} = \text{"}$$

$$= \frac{4n}{[\qquad]} = \frac{\tilde{E}_T}{E_I} e^{i(k-k_2)L}$$

$$\Rightarrow \boxed{\frac{\tilde{E}_T}{E_I} = \frac{4n\, e^{-i(k-k_2)L}}{\left[(n+1)^2 - (n-1)^2 e^{2ik_2L}\right]}}$$

R is reflection coeff., ratio of intensities of reflected + incident wave:

I of both are $\frac{1}{2}\epsilon_0 c|\tilde{E}|^2$ since they're in vacuum. So:

$$R = \left|\tilde{E}_R / E_I\right|^2$$

$$R = \frac{(n^2-1)^2 \left| e^{2ik_2L} - 1 \right|^2}{\left| (n+1)^2 - (n-1)^2 e^{2ik_2L} \right|^2}$$

The $\left| e^{2ik_2L} - 1 \right|^2 = \left(e^{2ik_2L} - 1 \right)\left(e^{-2ik_2L} - 1 \right)$

$$= 1 - e^{2ik_2L} - e^{-2ik_2L} + 1$$

$$= 2 - (\cos 2k_2L + i \sin(\,)) - (\cos(\,) - i \sin(\,))$$

$$= 2 - 2\cos 2k_2L = 2(1 - \cos 2k_2L)$$

The $\left| (n+1)^2 - (n-1)^2 e^{2ik_2L} \right|^2$

$$= \left[(n+1)^2 - (n-1)^2 e^{2ik_2L} \right]\left[(n+1)^2 - (n-1)^2 e^{-2ik_2L} \right]$$

$$= \underbrace{(n+1)^4 + (n-1)^4}_{''} - (n-1)^2(n+1)^2 e^{(\,)} - (n+1)^2(n-1)^2 e^{-(\,)}$$

$$= 2(n^4 + 6n^2 + 1) - (n-1)^2(n+1)^2 \big(\underbrace{e^{(\,)} + e^{-(\,)}}_{2\cos(\,)} \big)$$

$$= 2(n^4 + 6n^2 + 1) - (n^2-1)^2 \cdot 2\cos 2k_2L$$

$$\boxed{R = \frac{(n^2-1)^2 (1 - \cos 2k_2L)}{(n^4 + 6n^2 + 1) - (n^2-1)^2 \cos 2k_2L}}$$

$$T = \left| \frac{\tilde{E}_T}{\tilde{E}_I} \right|^2 = \frac{16 n^2}{\left| (n+1)^2 - (n-1)^2 e^{2ik_2 L} \right|^2}$$

$$\boxed{T = \frac{8n^2}{n^4 + 6n^2 + 1 - (n^2-1)^2 \cos 2k_2 L}}$$

Check that $R + T = 1$!

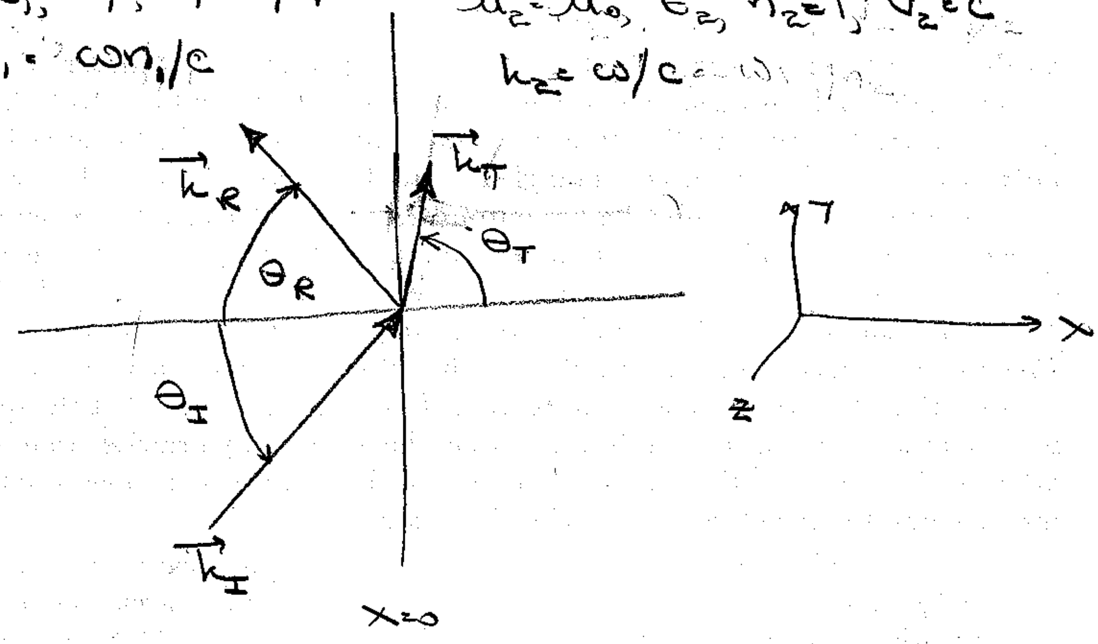
3

$\mu_1 = \mu_0,\ \epsilon_1,\ n_1,\ v_1 = c/n_1$

$k_1 = \omega/v_1 = \omega n_1/c$

$\mu_2 = \mu_0,\ \epsilon_2,\ n_2 = 1,\ v_2 = c$

$k_2 = \omega/c$



(a) Snell's Law: $n_2 \sin\theta_T = n_1 \sin\theta_I$

$$\underset{\sin\theta_T}{\underbrace{n_2\sin\theta_T}}$$

If $\theta_T = 90°$, then $n_1 \sin\theta_c = 1 \Rightarrow \boxed{\sin\theta_c = 1/n_1}$

At the critical angle θ_c, the transmitted wave propagates along the interface. For $\theta_I \geq \theta_c$, there is no transmitted wave; all the incident radiation is reflected. Thus θ_c is the critical incident angle for <u>total internal reflection</u>.

(b) The boundary conditions at $x=0$ require that

$$(k_T)_y = (k_I)_y = k_1 \sin\theta_I \longleftarrow \text{Snell's Law}$$

$$(k_T)_x^2 = \underset{\substack{\| \\ \frac{\omega^2}{c^2} = \frac{k_1^2}{n_1^2} = k_1^2 \sin^2\theta_c}}{k_2^2} - \overset{\substack{k_1^2\sin^2\theta_I \\ \|}}{(k_T)_y^2} = k_1^2 \underbrace{\left(\sin^2\theta_c - \sin^2\theta_I\right)}_{\theta_c \le \theta_I \Rightarrow \le 0} \le 0$$

So k_T is pure imaginary

$$(k_T)_x^2 = -k_1^2(\sin^2\theta_I - \sin^2\theta_c)$$

$$\therefore \boxed{(k_T)_x = i k_1 \sqrt{\sin^2\theta_I - \sin^2\theta_c} = i\frac{\omega}{c}\sqrt{n_1^2 \sin^2\theta_I - 1}}$$

choose + sign here because − sign leads to unphysical blowup of the fields in the vacuum region.

The fields in the vacuum region behave as

$$e^{i(\vec{k}_T \cdot \vec{r} - \omega t)} = e^{i((k_T)_y y + (k_T)_x x - \omega t)}$$

$$= \underbrace{e^{i(k_1 y \sin\theta_I - \omega t)}}_{\text{wave propagating in } y \text{ direction with wave number } \mathcal{K} \equiv k_1 \sin\theta_I \text{ and angular frequency } \omega.} \; \underbrace{e^{-\overbrace{k_1\sqrt{\sin^2\theta_I - \sin^2\theta_c}}^{\equiv \xi}\, x}}_{\text{exponential decay of the fields away from the interface}}$$

$$\boxed{\xi = k_1\sqrt{\sin^2\theta_I - \sin^2\theta_c}} \qquad \boxed{\mathcal{K} = (k_T)_y = k_1 \sin\theta_I}$$

Notice that

$$\vec{k}_T = \underset{\substack{\| \\ i\xi}}{(k_T)_x}\,\hat{e}_x + \underset{\substack{\| \\ k_1\sin\theta_I}}{(k_T)_y}\,\hat{e}_y = i\xi\,\hat{e}_x + k_1 \sin\theta_I\,\hat{e}_y$$

(c) Polarization $\perp$ to plane of incidence.

$$\vec{E}_T = \hat{e}_z \tilde{E}_0 e^{i(\vec{k}_T\cdot\vec{r} - \omega t)} \qquad \tilde{E}_0 = E_0 e^{i\delta}$$

$$\vec{B}_T = \frac{1}{\omega}\vec{k}_T \times \vec{E}_T$$

Maxwell equations in vacuum region

$$0 = \nabla\cdot\vec{E}_T \Rightarrow i\vec{k}_T\cdot\vec{E}_T = 0 \quad ✓$$

$$0 = \nabla\cdot\vec{B}_T \Rightarrow i\vec{k}_T\cdot\vec{B}_T = 0 \longleftarrow$$ This is true since $\vec{B}_T = \frac{1}{\omega}\vec{k}_T\times\vec{E}$ ✓

$$\nabla\times\vec{E}_T = -\frac{\partial\vec{B}_T}{\partial t} \Rightarrow i\vec{k}_T\times\vec{E}_T = +i\omega\vec{B}_T \Rightarrow \vec{B}_T = \frac{1}{\omega}\vec{k}_T\times\vec{E}_T \quad ✓$$

$$\nabla\times\vec{B}_T = \frac{1}{c^2}\frac{\partial\vec{E}_T}{\partial t} \Rightarrow i\vec{k}_T\times\vec{B}_T = -i\frac{\omega}{c^2}\vec{E}_T$$

$$\Rightarrow -\frac{\omega}{c^2}\vec{E}_T = \vec{k}_T\times\vec{B}_T = \frac{1}{\omega}\underbrace{\vec{k}_T\times(\vec{k}_T\times\vec{E}_T)}_{\vec{k}_T(\cancel{\vec{k}_T\cdot\vec{E}_T}) - \vec{E}_T(\vec{k}_T\cdot\vec{k}_T)}$$

$$\therefore \quad \frac{\omega^2}{c^2} = k_T^2 = \vec{k}_T\cdot\vec{k}_T = (k_T)_x^2 + (k_T)_y^2$$

(d) Polarization $\perp$ to plane of incidence

$$\vec{E}_T = \hat{e}_z \tilde{E}_0 e^{i(\vec{k}_T\cdot\vec{r} - \omega t)} \quad , \quad \tilde{E}_0 = E_0 e^{i\delta}$$

$$\boxed{\vec{E}_T = \hat{e}_z \tilde{E}_0 e^{i(k_1 y \sin\theta_I - \omega t)} e^{-\delta x}}$$

Faraday's law $\Longrightarrow$ $\cancel{i} \vec{k}_T \times \vec{E} = \cancel{i}\omega \vec{B}$

$$\vec{B}_T = \frac{1}{\omega}\vec{k}_T \times \vec{E}$$

$$= \frac{1}{\omega}\left(i\delta \hat{e}_x + k_1 \sin\theta_I \hat{e}_y\right) \times \hat{e}_z \tilde{E}_0 e^{i(k_1 y \sin\theta_1 - \omega t)} e^{-\delta x}$$

$$\boxed{\vec{B}_T = \left(-i\frac{\delta}{\omega}\hat{e}_y + \frac{k_1}{\omega}\sin\theta_I \hat{e}_x\right)\tilde{E}_0 e^{i(k_1 y \sin\theta_1 - \omega t)} e^{-\delta x}}$$

$$= \sqrt{\sin^2\theta_I - \sin^2\theta_c} \qquad \frac{n_1}{v_1} = \frac{n_1}{c_1}$$

The actual fields are obtained by taking the real parts:

$$\boxed{\begin{aligned}\vec{E}_T &= \hat{e}_z E_0 e^{-\delta x} \cos(k_1 y \sin\theta_I - \omega t + \delta) \\ \vec{B}_T &= \frac{k_1 \sin\theta_I}{\omega}\hat{e}_x E_0 e^{-\delta x} \cos(k_1 y \sin\theta_I - \omega t + \delta) \\ &\quad + \frac{\delta}{\omega}\hat{e}_y E_0 e^{-\delta x} \sin(k_1 y \sin\theta_I - \omega t + \delta)\end{aligned}}$$

Notice that ~~these~~ parts of $\vec{E}$ and $\vec{B}$ are just like a monochromatic plane wave propagating in the y direction, with wave number $k_1 \sin\theta_I$, angular frequency ω, phase velocity $\omega/k = \omega/k_1 \sin\theta_I = v_1 \sin\theta_I = c\sin\theta_I/n_1$, and linear polarization along $\hat{e}_z$

(d) Poynting vector:

$$\vec{S} = \frac{1}{\mu_0} \vec{E}_T \times \vec{B}_T$$

$$= \frac{1}{\mu_0} \left(\hat{e}_z E_0 e^{-\delta x} \cos(k_1 y \sin\theta_I - \omega t + \delta) \right)$$

$$\times \left(\frac{k_1 \sin\theta_I}{\omega} \hat{e}_x E_0 e^{-\delta x} \cos(k_1 y \sin\theta_I + \omega t + \delta) + \frac{\delta}{\omega} \hat{e}_y E_0 e^{-\delta x} \sin(k_1 y \sin\theta_I - \omega t + \delta) \right)$$

$$= \frac{1}{\mu_0} E_0^2 e^{-2\delta x} \left(\frac{k_1 \sin\theta_I}{\omega} \overbrace{\hat{e}_z \times \hat{e}_x}^{\hat{e}_y} \cos^2(k_1 y \sin\theta_I - \omega t + \delta) + \frac{\delta}{\omega} \underbrace{\hat{e}_z \times \hat{e}_y}_{-\hat{e}_x} \cos(k_1 y \sin\theta_I - \omega t + \delta) \times \sin(k_1 y \sin\theta_I - \omega t + \delta) \right)$$

$$\vec{S} = E_0^2 e^{-2\delta x} \left(\frac{k_1 \sin\theta_I}{\mu_0 \omega} \hat{e}_y \cos^2(k_1 y \sin\theta_I - \omega t + \delta) - \frac{\delta}{\mu_0 \omega} \hat{e}_x \underbrace{\cos(k_1 y \sin\theta_I - \omega t + \delta) \sin(k_1 y \sin\theta_I - \omega t + \delta)}_{= \frac{1}{2} \sin(2k_1 y \sin\theta_I - 2\omega t + 2\delta)} \right)$$

$= \frac{1}{2} \sin(2k_1 y \sin\theta_I - 2\omega t + 2\delta)$

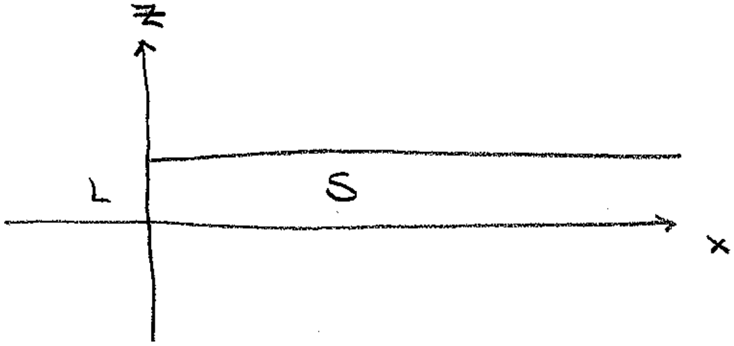
oscillation at frequency 2ω – 0 time average

Notice that the y-component of $\vec{B}_T$ does not contribute to the time-averaged Poynting vector

$$\boxed{\langle \vec{S} \rangle = \hat{e}_y \frac{k_1 \sin\theta_I}{2\mu_0 \omega} E_0^2 e^{-2\delta x}}$$

$$\frac{\sin\theta_I}{2\mu_0 v_1} = \frac{1}{2} v_1 \epsilon_1 \sin\theta_I$$

Energy is transported in the $+y$ direction with velocity $v_1 \sin\theta_I$.



The time-averaged power that passes through S is

$$P = \int \langle \vec{S} \rangle \cdot \underbrace{\vec{da}}_{\hat{e}_y \, dx \, dz}$$

$$= L \int_0^\infty \langle \vec{S} \rangle \cdot \hat{e}_y \, dx$$

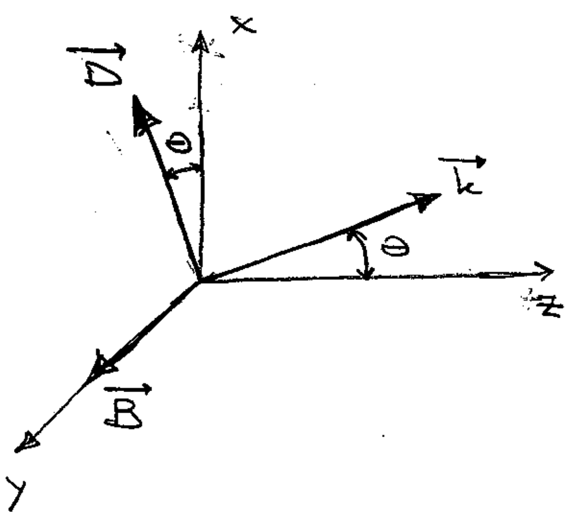
$$= L \, \frac{k_1 \sin\theta_1}{2\mu_0 \omega} E_0^2 \underbrace{\int_0^\infty dx \, e^{-2\xi x}}_{-\frac{e^{-2\xi x}}{2\xi}\Big|_0^\infty = \frac{1}{2\xi}}$$

$$\boxed{P = \frac{L}{2\xi} \underbrace{\frac{k_1 \sin\theta_1}{2\mu_0 \omega}}_{\frac{1}{2} v_1 \epsilon_1 \sin\theta_1} E_0^2}$$

4. $D_x = \epsilon_\perp E_x, \quad D_y = \epsilon_\perp E_y, \quad D_z = \epsilon_\parallel E_z$

$\vec{B} = \mu_0 \vec{H}$

$\vec{k} = k\underbrace{(\hat{e}_z \cos\theta + \hat{e}_x \sin\theta)}_{\hat{k}}$



$\vec{D} = \vec{\tilde{D}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$\vec{B} = \vec{\tilde{B}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$0 = \nabla\cdot\vec{D} = i\vec{k}\cdot\vec{D}$
$0 = \nabla\cdot\vec{B} = i\vec{k}\cdot\vec{B}$ $\Longrightarrow$ $\vec{D}$ and $\vec{B}$ are $\perp$ to $\vec{k}$

Assume a linearly polarized wave with
$\vec{\tilde{D}}_0 = D_0(\hat{e}_x \cos\theta - \hat{e}_z \sin\theta)$ and $\vec{\tilde{B}}_0 = B_0 \hat{e}_y$

or

$$\boxed{\begin{aligned} \vec{D} &= D_0(\hat{e}_x \cos\theta - \hat{e}_z \sin\theta)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \\ \vec{B} &= B_0 \hat{e}_y\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \end{aligned}}$$

(a) $$\boxed{\vec{E} = D_0\left(\frac{1}{\epsilon_\perp}\hat{e}_x \cos\theta - \frac{1}{\epsilon_\parallel}\hat{e}_z \sin\theta\right) e^{i(\vec{k}\cdot\vec{r} - \omega t)}}$$

(b) $\nabla\times\vec{B} = i\vec{k}\times\hat{e}_y B_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} = -ik(\hat{e}_x \cos\theta - \hat{e}_z \sin\theta) B_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$\vec{k}\times\hat{e}_y \longrightarrow k\,\hat{k}\times\hat{e}_y = k(\hat{e}_z\times\hat{e}_y \cos\theta + \hat{e}_x\times\hat{e}_y \sin\theta) = k(-\hat{e}_x \cos\theta + \hat{e}_z \sin\theta)$

$\|$

$\mu_0 \frac{\partial \vec{D}}{\partial t} = -i\omega\mu_0 D_0(\hat{e}_x \cos\theta - \hat{e}_z \sin\theta)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$$\Longrightarrow \quad k B_o = \omega \mu_o D_o \quad \Longrightarrow \quad \boxed{B_o = \frac{\omega}{k} \mu_o D_o}$$

(c) $\nabla \times \vec{E} = i\vec{k} \times \vec{E} = ik \underbrace{\hat{k} \times \left(\frac{1}{\epsilon_\perp} \hat{e}_x \cos\theta - \frac{1}{\epsilon_{\parallel}} \hat{e}_z \sin\theta \right)} D_o e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$$= (\hat{e}_z \cos\theta + \hat{e}_x \sin\theta) \times \left(\frac{1}{\epsilon_\perp} \hat{e}_x \cos\theta - \frac{1}{\epsilon_{\parallel}} \hat{e}_z \sin\theta \right)$$

$$= \frac{1}{\epsilon_\perp} \underbrace{\hat{e}_z \times \hat{e}_x}_{\hat{e}_y} \cos^2\theta - \frac{1}{\epsilon_{\parallel}} \underbrace{\hat{e}_x \times \hat{e}_z}_{-\hat{e}_y} \sin^2\theta$$

$$= \hat{e}_y \left(\frac{1}{\epsilon_\perp} \cos^2\theta + \frac{1}{\epsilon_{\parallel}} \sin^2\theta \right)$$

$$\nabla \times \vec{E} = ik \hat{e}_y \left(\frac{1}{\epsilon_\perp} \cos^2\theta + \frac{1}{\epsilon_{\parallel}} \sin^2\theta \right) D_o e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\parallel$$

$$-\frac{\partial \vec{B}}{\partial t} = i\omega B_o \hat{e}_y e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\Longrightarrow \quad \omega B_o = k \left(\frac{1}{\epsilon_\perp} \cos^2\theta + \frac{1}{\epsilon_{\parallel}} \sin^2\theta \right) D_o$$

$$\Longrightarrow \quad \boxed{B_o = \frac{k}{\omega} \left(\frac{1}{\epsilon_\perp} \cos^2\theta + \frac{1}{\epsilon_{\parallel}} \sin^2\theta \right) D_o}$$

(d) $\dfrac{B_o}{D_o} = \dfrac{\omega}{k} \mu_o = \dfrac{k}{\omega} \left(\dfrac{1}{\epsilon_\perp} \cos^2\theta + \dfrac{1}{\epsilon_{\parallel}} \sin^2\theta \right)$

$$\Longrightarrow \quad \boxed{\begin{array}{l} \dfrac{\omega^2}{k^2} = \dfrac{1}{\mu_o} \left(\dfrac{1}{\epsilon_\perp} \cos^2\theta + \dfrac{1}{\epsilon_{\parallel}} \sin^2\theta \right) = V^2 \\ V = \sqrt{\dfrac{1}{\mu_o} \left(\dfrac{1}{\epsilon_\perp} \cos^2\theta + \dfrac{1}{\epsilon_{\parallel}} \sin^2\theta \right)} \end{array}}$$

$$\boxed{B_0 = \frac{\omega}{k}\mu_0 D_0 = \mu_0 v D_0 = \sqrt{\mu_0\left(\frac{1}{\epsilon_\perp}\cos^2\theta + \frac{1}{\epsilon_\parallel}\sin^2\theta\right)}\, D_0}$$

Special cases:

(i) $\epsilon_\parallel = \epsilon_\perp = \epsilon$: $v = \frac{1}{\sqrt{\mu_0 \epsilon}}$ ✓ Propagation in an isotropic medium.

$$B_0 = \mu_0 v D_0 = \mu_0 \epsilon v E_0 = \sqrt{\mu_0 \epsilon}\, E_0 = \frac{1}{v} E_0 \checkmark$$

(ii) $\theta = 0$: propagation along the z axis. $\vec{D}$ and $\vec{E}$ have only x components, so only $\epsilon_\perp$ matters. This case should be like propagation in an isotropic medium with permittivity $\epsilon_\perp$.

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_\perp}} \checkmark$$

$$B_0 = \mu_0 v D_0 = \mu_0 \frac{1}{\sqrt{\mu_0 \epsilon_\perp}} \epsilon_\perp E_0 = \sqrt{\mu_0 \epsilon_\perp}\, E_0 = \frac{1}{v} E_0 \checkmark$$

(iii) $\theta = 90°$: propagation along x axis. $\vec{D}$ and $\vec{E}$ have only z components, so only $\epsilon_\parallel$ matters. This case should be like propagation in an isotropic medium with permittivity $\epsilon_\parallel$

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_\parallel}} \checkmark$$

$$B_0 = \mu_0 v D_0 = \mu_0 \frac{1}{\sqrt{\mu_0 \epsilon_\parallel}} \epsilon_\parallel E_0 = \sqrt{\mu_0 \epsilon_\parallel}\, E_0 = \frac{1}{v} E_0 \checkmark$$

(e) $u = \frac{1}{2}(\vec{D}\cdot\vec{E} + \vec{B}\cdot\vec{H})$

$$\vec{D}\cdot\vec{E} = \frac{1}{\epsilon_\parallel} D_z^2 + \frac{1}{\epsilon_\perp}(D_x^2 + D_y^2)$$

$$= \left(\frac{1}{\epsilon_\parallel}\sin^2\theta + \frac{1}{\epsilon_\perp}\cos^2\theta\right) D_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t)$$

$$\langle \vec{D}\cdot\vec{E}\rangle = \frac{1}{2}\left(\frac{1}{\epsilon_\parallel}\sin^2\theta + \frac{1}{\epsilon_\perp}\cos^2\theta\right) D_0^2$$

$$\vec{B}\cdot\vec{H} = \frac{1}{\mu_0} B^2 = \frac{1}{\mu_0} B_0^2 \cos^2(\vec{k}\cdot\vec{r} - \omega t)$$

$$\langle \vec{B}\cdot\vec{H}\rangle = \frac{1}{2\mu_0} B_0^2 = \frac{1}{2\mu_0}\underbrace{\left(\frac{\omega}{k}\right)^2}_{v^2 = \frac{1}{\mu_0}\left(\frac{1}{\epsilon_\perp}\cos^2\theta + \frac{1}{\epsilon_\parallel}\sin^2\theta\right)} \mu_0^2 D_0^2 = \frac{1}{2}\left(\frac{1}{\epsilon_\perp}\cos^2\theta + \frac{1}{\epsilon_\parallel}\sin^2\theta\right) D_0^2$$

Electric and magnetic fields have the same energy density.

$$\boxed{\langle u\rangle = \frac{1}{2}\left(\frac{1}{\epsilon_\perp}\cos^2\theta + \frac{1}{\epsilon_\parallel}\sin^2\theta\right) D_0^2}$$

(i) $\epsilon_\perp = \epsilon_\parallel = \epsilon$, $u = \frac{1}{2\epsilon} D_0^2 = \frac{1}{2}\epsilon E_0^2$

(ii) $\theta = 0$: $u = \frac{1}{2\epsilon_\perp} D_0^2 = \frac{1}{2}\epsilon_\perp E_0^2$

(iii) $\theta = 90°$: $u = \frac{1}{2\epsilon_\parallel} D_0^2 = \frac{1}{2}\epsilon_\parallel E_0^2$

$$\vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

$$= \frac{1}{\mu_0} D_0 B_0 \underbrace{\left(\frac{1}{\epsilon_\perp}\hat{e}_x\cos\theta - \frac{1}{\epsilon_\parallel}\hat{e}_z\sin\theta\right)\times\hat{e}_y}\cos^2(\vec{k}\cdot\vec{r} - \omega t)$$

$$= \frac{1}{\epsilon_\perp}\underbrace{\hat{e}_x\times\hat{e}_y}_{\hat{e}_z}\cos\theta - \frac{1}{\epsilon_\parallel}\underbrace{\hat{e}_z\times\hat{e}_y}_{-\hat{e}_x}\sin\theta$$

$$= \frac{1}{\epsilon_\perp}\hat{e}_z\cos\theta + \frac{1}{\epsilon_\parallel}\hat{e}_x\sin\theta$$

$$\langle \vec{S} \rangle = \frac{1}{2\cancel{\mu_0}} D_0 \underbrace{\cancel{\mu_0} v D_0}_{B_0} \left(\frac{1}{\epsilon_\perp} \hat{e}_z \cos\theta + \frac{1}{\epsilon_\parallel} \hat{e}_x \sin\theta \right)$$

$$\boxed{\langle \vec{S} \rangle = \frac{1}{2} v D_0^2 \left(\frac{1}{\epsilon_\perp} \hat{e}_z \cos\theta + \frac{1}{\epsilon_\parallel} \hat{e}_x \sin\theta \right)}$$

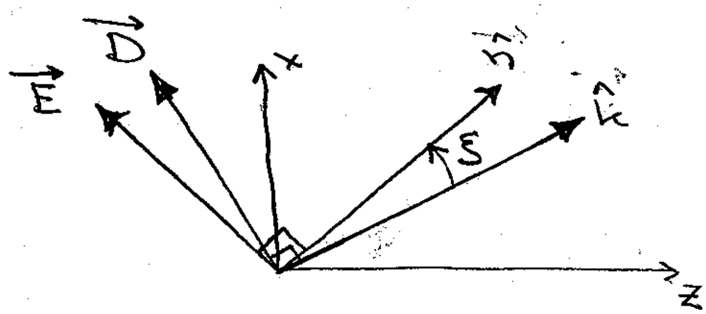
$$\boxed{\langle \vec{S} \rangle = \underbrace{\frac{1}{2} v D_0^2 A}_{\text{intensity } I} \hat{n}}$$

Define the unit vector

$$\boxed{\hat{n} = \frac{\frac{1}{\epsilon_\perp} \hat{e}_z \cos\theta + \frac{1}{\epsilon_\parallel} \hat{e}_x \sin\theta}{\left(\frac{1}{\epsilon_\perp^2} \cos^2\theta + \frac{1}{\epsilon_\parallel^2} \sin^2\theta \right)^{1/2}}}$$

$\equiv A$

Energy propagates in the $\hat{n}$ direction



$\hat{n}$ shifts in direction shown for $\epsilon_\perp > \epsilon_\parallel$; oppositely for $\epsilon_\perp < \epsilon_\parallel$.

$\vec{D} \perp \hat{k}$ and $\vec{E} \perp \hat{n}$ (since $\vec{S} = \vec{E} \times \vec{H}$)

We can define the velocity at which the energy propagates along $\hat{n}$ by

$$v_E = \frac{\langle I \rangle}{\langle U \rangle} = \frac{vA}{\left(\frac{1}{\epsilon_\perp} \cos^2\theta + \frac{1}{\epsilon_\parallel} \sin^2\theta \right)} = \frac{A}{\sqrt{\mu_0 \left(\frac{1}{\epsilon_\perp} \cos^2\theta + \frac{1}{\epsilon_\parallel} \sin^2\theta \right)}}$$

But notice that

$$\cos\delta = \hat{n} \cdot \hat{k} = \frac{1}{A} \left(\frac{1}{\epsilon_\perp} \cos^2\theta + \frac{1}{\epsilon_\parallel} \sin^2\theta \right)$$

$$v_E = \frac{v}{\cos\delta} \Rightarrow \boxed{v = v_E \cos\delta}$$

Get $v\hat{k}$ by projecting $v_E \hat{n}$ onto $\hat{k}$ axis.

This means there is no difference between saying that the energy propagates along $\hat{n}$ at speed v_E and saying that it propagates along $\hat{k}$ at speed v.