Homework #6 *due date is Wednesday 10/15/2014* ***in my mailbox by 5PM***
(see syllabus about late HWs)
(Note that all Griffiths Problems are from 4th Ed.)

1) A problem on Birefringence. Consider a linear, homogeneous, nonconducting, but anisotropic dielectric medium in which

$$D_x = \epsilon_x E_x \ , \qquad D_y = \epsilon_y E_y \ , \qquad D_z = \epsilon_z E_z \ .$$

Such a medium, called *birefringent*, is used to make quarter- and half-wave plates.

In this problem we are interested in monochromatic plane waves with frequency ω that propagate in the $+x$ direction. Throughout this problem we use phasors, remembering that the actual electric and magnetic fields are given by the real parts of the phasors. You do not have to take the real part unless you want to.

(a) A wave that is linearly polarized along the y axis, i.e., a wave with electric field

$$\mathbf{E} = \hat{\mathbf{e}}_y E_0 e^{i(kx-\omega t)} \ ,$$

is a solution of the Maxwell equations. *Derive* the associated magnetic field $\mathbf{B}$ and the phase velocity v_y of this wave.

(b) A wave that is linearly polarized along the z axis, i.e., a wave with electric field

$$\mathbf{E} = \hat{\mathbf{e}}_z E_0 e^{i(kx-\omega t)} \ ,$$

is a solution of the Maxwell equations. *Give* the associated magnetic field $\mathbf{B}$ and the phase velocity v_z of this wave. Don't do any work on this part if you can simply translate your results for part (a).

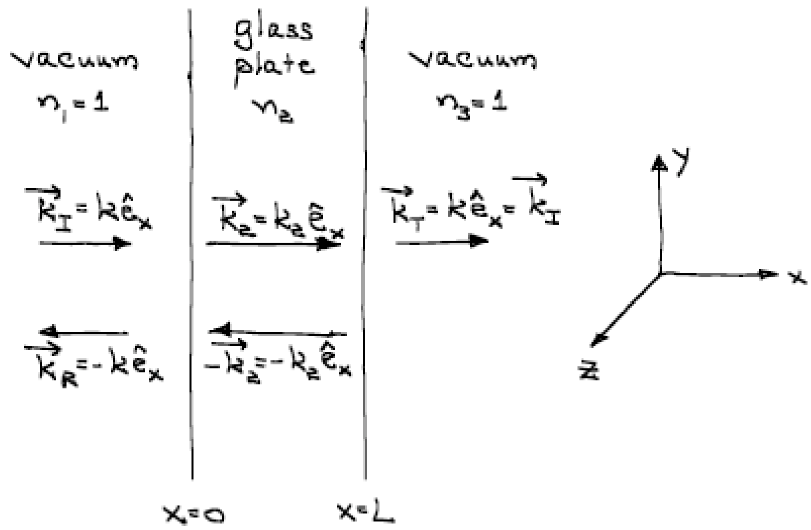
(c) Suppose that $x = 0$ is the left boundary of the dielectric medium: the medium occupies the region $x > 0$, and the region $x < 0$ is vacuum. A wave incident on the medium from the vacuum is transmitted into the medium. Assume that at $x = 0$ the wave propagating into the medium is linearly polarized along the unit vector

$$\frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z) \ .$$

Write an expression for the electric field $\mathbf{E}$ at all $x > 0$. (Hint: Use the superposition principle.)

(d) Now assume that $v_y > v_z$. (i) *Find* the smallest value of x in the medium such that the wave has right-circular polarization. (ii) *Find* the smallest value of x in the medium such that the wave is linearly polarized along the unit vector $(\hat{\mathbf{e}}_y - \hat{\mathbf{e}}_z)/\sqrt{2}$ (i.e., orthogonal to the linear polarization at $x = 0$). (iii) *Find* the smallest value of x in the medium such that the wave has left-circular polarization. If the medium has a right boundary at the position found in (i), it is called a *quarter-wave plate*; if the medium has a right boundary at the position found in (ii), it is called a *half-wave plate*. Can you explain the origin of these terms?

2) A problem on Fabry-Perot cavities. Consider a plate of glass of thickness L, as shown in the drawing below. The glass has electric permittivity $\epsilon_2 \equiv \epsilon$, magnetic permeability $\mu_2 = \mu_0$, and index of refraction $n_2 = \sqrt{\epsilon_2/\epsilon_0} \equiv n$. Within the glass plate, electromagnetic waves have a phase velocity $v_2 = c/n \equiv v$.



A plane electromagnetic wave, linearly polarized in the y direction, is incident normally on the glass plate from the vacuum region on the left; the electric and magnetic fields of the incident wave ($x < 0$) are given by

$$\mathbf{E}_I = E_I \hat{\mathbf{e}}_y e^{i(kx-\omega t)} , \qquad \mathbf{B}_I = \frac{1}{c} E_I \hat{\mathbf{e}}_z e^{i(kx-\omega t)} ,$$

where $\omega = ck$ and E_I is a real amplitude.

Some of the incident wave is reflected, and some is transmitted through to the vacuum region on the right side of the glass plate. The electric and magnetic fields of the reflected wave ($x < 0$) are given by

$$\mathbf{E}_R = \tilde{E}_R \hat{\mathbf{e}}_y e^{-i(kx+\omega t)} , \qquad \mathbf{B}_R = -\frac{1}{c} \tilde{E}_R \hat{\mathbf{e}}_z e^{-i(kx+\omega t)} ,$$

and the electric and magnetic fields of the transmitted wave ($x > L$) are given by

$$\mathbf{E}_T = \tilde{E}_T \hat{\mathbf{e}}_y e^{i(kx-\omega t)} , \qquad \mathbf{B}_T = \frac{1}{c} \tilde{E}_T \hat{\mathbf{e}}_z e^{i(kx-\omega t)} .$$

Inside the glass plate there are both left-going and right-going waves. The electric and magnetic fields of the right-going wave ($0 < x < L$) are given by

$$\mathbf{E}_2^{(R)} = \tilde{C} \hat{\mathbf{e}}_y e^{i(k_2 x-\omega t)} , \qquad \mathbf{B}_2^{(R)} = \frac{1}{v} \tilde{C} \hat{\mathbf{e}}_z e^{i(k_2 x-\omega t)} ,$$

where $\omega = vk$, and the electric and magnetic fields of the left-going wave $(0 < x < L)$ are given by

$$\mathbf{E}_2^{(L)} = \tilde{D}\hat{\mathbf{e}}_y e^{-i(k_2 x + \omega t)} , \qquad \mathbf{B}_2^{(L)} = -\frac{1}{v}\tilde{D}\hat{\mathbf{e}}_z e^{-i(k_2 x + \omega t)} .$$

(a) The boundary conditions at the left side of the glass plate $(x = 0)$ give two relations among E_I, $\tilde{E}_R$, $\tilde{C}$, and $\tilde{D}$. *Find* these two relations.

(b) The boundary conditions at the right side of the glass plate $(x = L)$ give two relations among $\tilde{E}_T$, $\tilde{C}$, and $\tilde{D}$. *Find* these two relations.

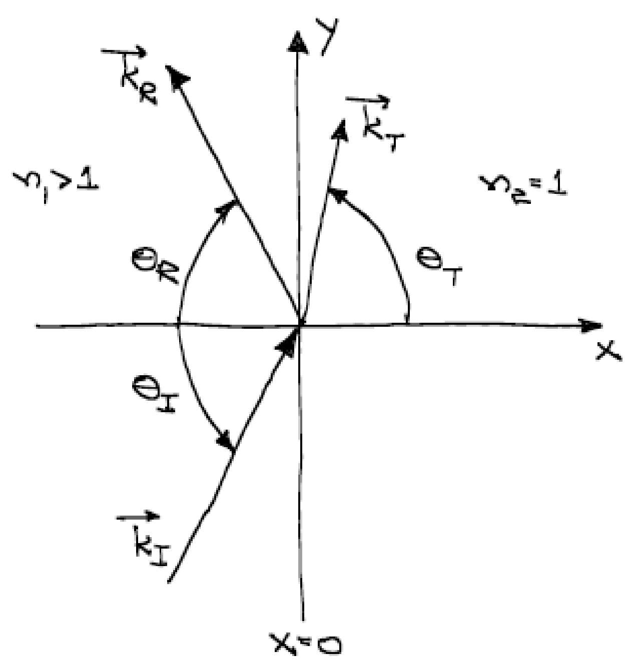
(c) *Show* that when L is an integral number of half-wavelengths (within the glass), all of the wave is transmitted, and none is reflected.

(d) *Show* that when L is an integral number of half-wavelengths plus a quarter-wavelength (within the glass), the reflected and transmitted complex amplitudes are given by

$$\tilde{E}_R = \frac{1-n^2}{1+n^2}E_I , \qquad \tilde{E}_T = \frac{2n}{1+n^2}E_I e^{-i(k-k_2)L} .$$

(e) For an arbitrary width of the plate, *find* the ratios $\tilde{E}_R/E_I$ and $\tilde{E}_T/E_I$. *Find* the reflection and transmission coefficients, R and T, i.e., the ratios of reflected and transmitted intensities to incident intensity, and *show* that energy is conserved. In doing the algebra for this problem, you might it useful to introduce $\tilde{F}_T \equiv \tilde{E}_T e^{i(k-k_2)L}$; $\tilde{F}_T$ is just a different complex amplitude for the transmitted wave, which differs from $\tilde{E}_T$ by a wavelength-dependent phase.

3) Problem on total internal reflection. A dielectric medium with index of refraction $n_1 > 1$ occupies the region $x < 0$, with vacuum in the region $x > 0$. A monochromatic plane wave, with angular frequency ω and wave number $k_1 = \omega/v_1 = \omega n_1/c$, is incident on the plane interface $(x = 0)$ between the two regions.



(a) What is the critical angle of incidence, $\theta_I = \theta_c$, such that the transmitted wave is refracted to $\theta_T = 90°$?

In the remainder of this problem we are interested in what happens when $\theta_I > \theta_c$, a situation called *total internal reflection* because there is no wave transmitted into the vacuum region and all the incident radiation is reflected. You experience the effects of total internal reflection when you look into an aquarium. Even though there is no transmitted wave, there are electric and magnetic fields on the vacuum side of the interface. This problem explores the behavior of these fields.

(b) Use the boundary condition that gives Snell's law to *find* the components of the wave vector $\mathbf{k}_T$ in the vacuum region and thereby to *show* that the fields in the vacuum region behave as

$$e^{-\xi x} e^{i(\kappa y - \omega t)} \, .$$

Find the decay constant ξ and the effective wave number κ. Write your answers in terms of k_1, θ_c, and θ_I. (Hint: You should be prepared to allow one of the components of $\mathbf{k}_T$ to be imaginary.)

The fields in the vacuum region are called an *evanescent wave* because they decay in the direction normal to the interface. For the remainder of the problem, suppose the incident wave is polarized perpendicular to the plane of incidence. Write the electric and magnetic fields in the vacuum region as

$$\mathbf{E}_T = \hat{\mathbf{e}}_z \tilde{E}_0 e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)} \, ,$$
$$\mathbf{B}_T = \frac{1}{\omega} \mathbf{k}_T \times \mathbf{E}_T \, ,$$

where $\tilde{E}_0 = E_0 e^{i\delta}$.

(c) *Verify* that these fields satisfy the Maxwell equations provided that $\omega^2/c^2 \equiv k_2^2 = \mathbf{k}_T \cdot \mathbf{k}_T = (k_T)_x^2 + (k_T)_y^2$. (Hint: This is straightforward. I just want you to reassure you that nothing has gone wrong because $\mathbf{k}_T$ is now complex. In particular, you might have thought that you should have $\omega^2/c^2 = \mathbf{k}_T^* \cdot \mathbf{k}_T$, since $\mathbf{k}_T^* \cdot \mathbf{k}_T$ is the squared magnitude of a complex vector, but this part shows that $\omega^2/c^2 = \mathbf{k}_T \cdot \mathbf{k}_T$.)

(d) *Determine* the real electric and magnetic fields in the vacuum region. Write your answer in terms of the real amplitude E_0 and the phase δ. You don't need to relate the amplitude and phase of the evanescent wave to the amplitude and phase of the incident wave.

(e) *Find* the instantaneous and time-averaged Poynting vector in the vacuum region. In what direction is energy transported in the vacuum region. *Find* the average total power that passes in the $+y$ direction through the surface defined by $0 < z < L$ and $x > 0$.

4) EXTRA CREDIT: Consider a lossless, source-free dielectric medium that has magnetic permeability μ_0. The medium has anisotropic dielectric properties described by two permittivities, $\epsilon_\perp$ and $\epsilon_\parallel$. In an appropriate Cartesian coordinate system the relation between the $\mathbf{D}$ and $\mathbf{E}$ fields is

$$D_x = \epsilon_\perp E_x \,, \qquad D_y = \epsilon_\perp E_y \,, \qquad D_z = \epsilon_\parallel E_z \,.$$

Thus $\epsilon_\perp$ is the permittivity along the x and y axes, and $\epsilon_\parallel$ is the permittivity along the z axis.

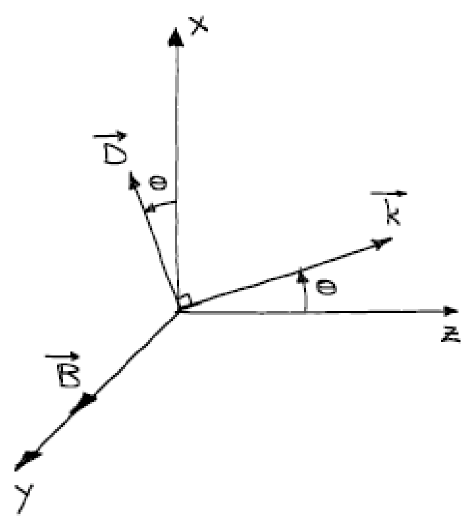
We want to solve for a particular plane wave with angular frequency ω propagating in this anisotropic medium. We choose the wave vector to be

$$\mathbf{k} = k\hat{\mathbf{k}} = k(\hat{\mathbf{e}}_z \cos\theta + \hat{\mathbf{e}}_x \sin\theta) \,,$$

where k is the wave number and the propagation direction $\hat{\mathbf{k}}$ makes an angle θ with the z axis. The Maxwell equations $\nabla \cdot \mathbf{D} = 0$ and $\nabla \cdot \mathbf{B} = 0$ tell us that $\mathbf{D}$ and $\mathbf{B}$ are orthogonal to $\mathbf{k}$. Consistent with these facts, we assume the following linearly polarized phasor forms for $\mathbf{D}$ and $\mathbf{B}$:

$$\mathbf{D} = D_0(\hat{\mathbf{e}}_x \cos\theta - \hat{\mathbf{e}}_z \sin\theta)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \,, \qquad \mathbf{B} = B_0\hat{\mathbf{e}}_y e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \,.$$

In these expressions D_0 and B_0 are real amplitudes, and we realize that we must actually take the real parts of the phasors. The situation is pictured in the diagram below.



(a) *Write* the phasor expression for $\mathbf{E}$.

(b) Use the Maxwell equation

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$

to *find* a relation between B_0 and D_0.

(c) Use the Maxwell equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

to *find* a second relation between B_0 and D_0.

(d) Combine the results of parts (b) and (c) to *find* the relation between D_0 and B_0 and the phase velocity v in terms of μ_0, $\epsilon_\perp$, $\epsilon_\parallel$, and θ. *Examine* the following special cases: (i) $\epsilon_\parallel = \epsilon_\perp = \epsilon$, (ii) $\theta = 0$ (propagation along the z axis), and (iii) $\theta = 90°$ (propagation along the x axis). *Interpet* your results for these special cases.

(e) In a linear, but anisotropic medium the electromagnetic energy density and Poynting vector are given by

$$U = \frac{1}{2}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H}) , \qquad \mathbf{S} = \mathbf{E}\times\mathbf{H} .$$

Find the time-averaged energy density and Poynting vector associated with the wave in terms of $\epsilon_\perp$, $\epsilon_\parallel$, θ, and D_0. In what direction does the energy propagate?