1) $\vec{E}(z,t) = \vec{E}_0 \cos(\sqrt{6}\, z - (6\times10^{10})\, t)$

cm   seconds

a) Frequency $\nu = ?$ $\quad \omega = 6\times10^{10}\ s^{-1}$

So $\boxed{\nu = \omega/2\pi = 9.5\times10^{9}\ Hz}$ ← Microwave

b) $n = ?$ $\quad v = \frac{\omega}{k}$, $k = \sqrt{6}\ cm^{-1}$, $v = \sqrt{6}\times10^{10}\ \frac{cm}{s}$

So $v = \frac{c}{n}$ and $\boxed{n = \frac{c}{v} = \frac{ck}{\omega} = 1.22}$

c) $\lambda_{vac} = \frac{c}{\nu} = \frac{2\pi c}{\omega} = \boxed{3.1\ cm}$

$\lambda_{medium} = \frac{\lambda_{vac}}{n} = \frac{2\pi}{k} = \boxed{2.56\ cm}$

also

d) $v\vec{B} = \hat{k}\times\vec{E}$

$\rightarrow \vec{B} = \hat{k}\times\frac{\vec{E}}{v} = \hat{z}\times\frac{\vec{E}}{c/n}$

$$\boxed{\vec{B} = n\,\hat{z}\times\frac{\vec{E}_0}{c}\cos(\sqrt{6}\, z - (6\times10^{10})\, t)}$$

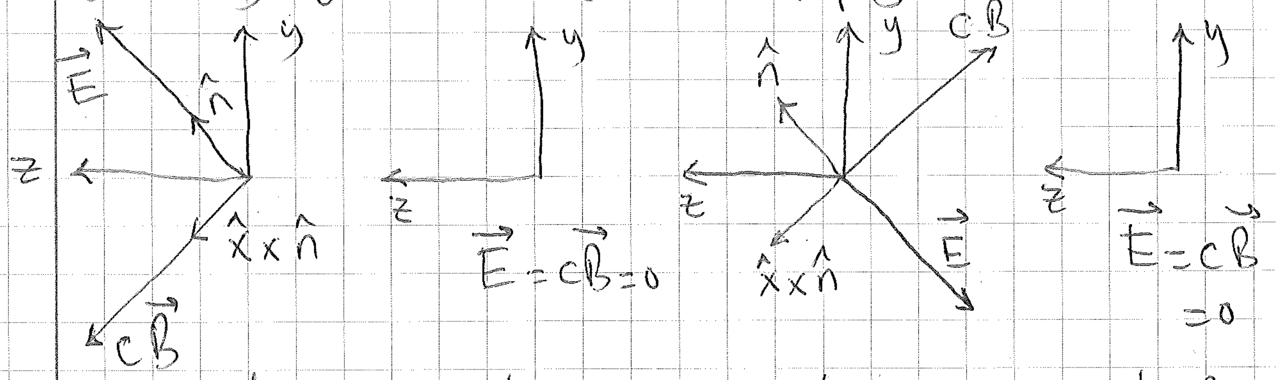
2) a) Linear Polar. $\vec{\tilde{E}}_0 = E_0 \hat{n}$ ← $\perp$ to $\hat{x}$

$\vec{E} = Re(E_0 \hat{n} e^{i(kx-\omega t)})$

$$\boxed{\begin{aligned} \vec{E} &= \hat{n} E_0 \cos(kx-\omega t) \\ \vec{B} &= Re\left(\frac{E_0}{c}(\hat{x}\times\hat{n})e^{i(kx-\omega t)}\right) \\ \vec{B} &= \hat{x}\times\hat{n}\,\frac{E_0}{c}\cos(kx-\omega t) \end{aligned}}$$

At $x=0$, $\vec{E} = \hat{n} E_0 \cos\omega t$
$\vec{B} = \hat{x} \times \hat{n} \frac{E_0}{c} \cos\omega t$ } In phase

Drawings for $x=0$: ($\hat{x}$ out of page)



$\omega t = 0$ $\quad\omega t = \frac{\pi}{2}$ $\quad\omega t = \pi$ $\quad\omega t = \frac{3\pi}{2}$

(b) Energy density $u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2$

$$\boxed{\text{instantaneous: } u = \epsilon_0 E_0^2 \cos^2(kx-\omega t)}$$

Time avg: $\langle u \rangle = \epsilon_0 E_0^2 \underbrace{\langle \cos^2(kx-\omega t)\rangle}_{\frac{1}{2} \text{ from Class}}$

$$\boxed{\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2}$$

We can also show this using Phasors (HW 6)

$\langle \vec{E}\cdot\vec{E} \rangle = \frac{1}{2}\langle \tilde{\vec{E}}^* \tilde{\vec{E}} \rangle = \frac{1}{2} E_0^2$, etc for $\langle \vec{B}\cdot\vec{B} \rangle$

Poynting, $\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$

$$= \underbrace{\hat{n} \times (\hat{x} \times \hat{n})}_{\hat{x}\ !} \frac{1}{\mu_0 c} E_0^2 \cos^2(kx-\omega t)$$

$\mu_0 c \to c\epsilon_0$

$$\boxed{\vec{S} = \hat{x}\, c\epsilon_0 E_0^2 \cos^2(kx-\omega t) = \hat{x}\, c u}$$

$$\boxed{\langle \vec{S} \rangle = \hat{x}\,\frac{1}{2} c\epsilon_0 E_0^2}$$ Show this using phasors too!

c) Circular Polar. $\tilde{\vec{E}}_0 = E_0(\hat{y} \pm i\hat{z})$

$$\tilde{\vec{B}}_0 = \frac{\hat{x}}{c} \times \tilde{\vec{E}}_0 = \frac{E_0}{c}(\hat{z} \mp i\hat{y})$$

Real parts:

$$\vec{E} = \mathrm{Re}\left[E_0(\hat{y} \pm i\hat{z})e^{i(kx-\omega t)}\right]$$

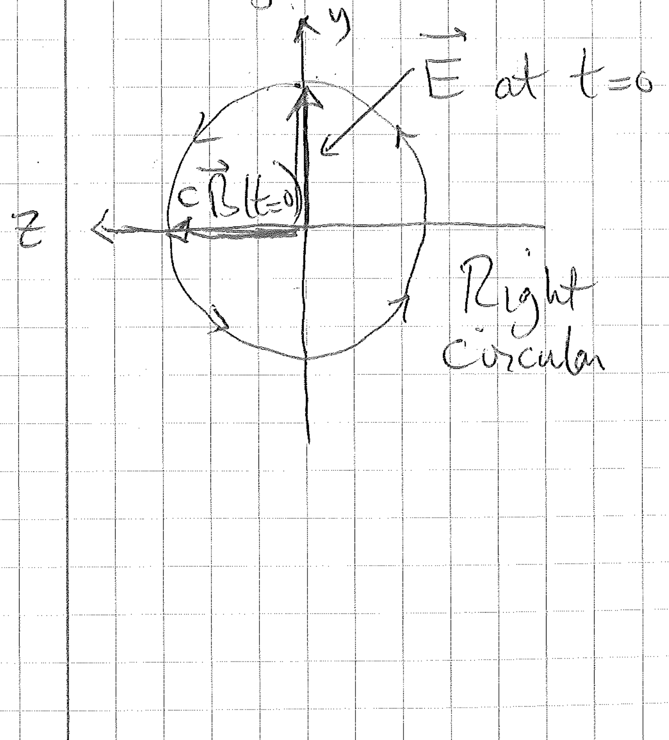
$$\vec{E} = E_0\left[\hat{y}\cos(kx-\omega t) \mp \hat{z}\sin(kx-\omega t)\right]$$
$$\vec{B} = \frac{E_0}{c}\left[\hat{z}\cos(kx-\omega t) \pm \hat{y}\sin(kx-\omega t)\right]$$

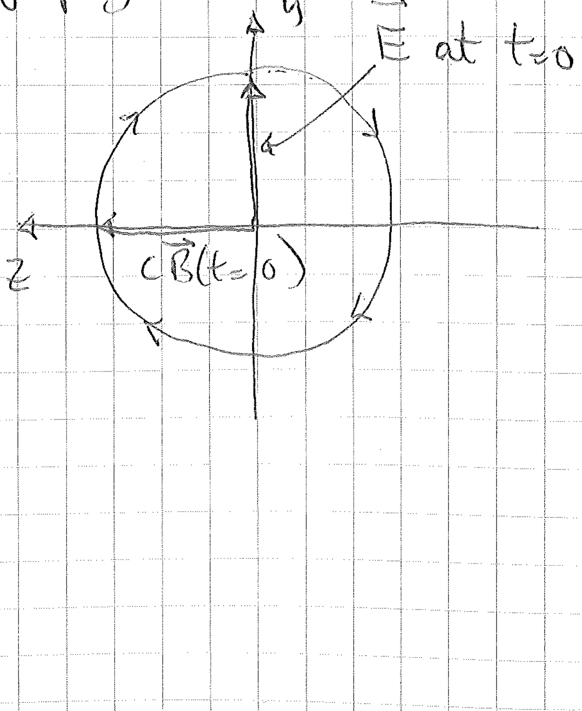
At $\underline{x=0}$ (upper sign is Right circular, lower sign is Left Circular)

$$\vec{E} = E_0(\hat{y}\cos\omega t \pm \hat{z}\sin\omega t)$$
$$\vec{B} = \frac{E_0}{c}(\hat{z}\cos\omega t \mp \hat{y}\sin\omega t)$$

(because $\sin(-\omega t) = -\sin\omega t$)

Drawings with $\hat{x}$ out of page





d) $u = \frac{1}{2}\epsilon_0 (\underbrace{E^2}_{=E_0^2} + c^2 \underbrace{B^2}_{B_0^2}) = = \boxed{\epsilon_0 E_0^2}$ from $\underbrace{c^2 B_0^2}_{=E_0^2}$

Energy density is constant in space & time

So time-avg is same also:

$$\boxed{\langle u \rangle = \epsilon_0 E_0^2}$$

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0 c} E_0^2 (\overbrace{\hat{y} \times \hat{z}}^{\hat{x}} \cos^2 \omega t - \underbrace{\hat{z} \times \hat{y}}_{-\hat{x}} \sin^2 \omega t)$$

$$\boxed{\begin{aligned} \vec{S} &= \hat{x} c \epsilon_0 E_0^2 \\ \vec{S} &= \hat{x} c u \end{aligned}} \rightarrow \langle \vec{S} \rangle = \vec{S}$$

Why are the average quantities 2x those of linear polarization?

3) Standing waves: $\vec{E}_1 = \hat{x} E_0 \cos(kz - \omega t)$

$\vec{E}_2 = \hat{x} E_0 \cos(kz + \omega t)$

a) $\vec{E}_3 = \vec{E}_1 + \vec{E}_2$

$= ? = \hat{x} E_0 [\cos(kz)\cos(\omega t) + \sin(kz)\sin(\omega t) + \cos(kz)\cos(\omega t) - \sin(kz)\sin(\omega t)]$

$$\boxed{\vec{E}_3 = \hat{x}\, 2E_0 \cos(kz)\cos(\omega t)}$$

You should also show this using phasors!

b) $\vec{B}_3 = ? = \frac{\vec{k}}{c} \times \vec{E}_3$? No, $\vec{B}_3$ is NOT a ~~travelling~~ wave!

Using Phasors: $\vec{B}_3 = \mathrm{Re}\left[\underbrace{\tilde{\vec{B}}_3(z)\, e^{-i\omega t}}_{\tilde{\vec{B}}_3(z,t)}\right]$

and $\vec{E}_3 = \mathrm{Re}\left[\underbrace{\tilde{\vec{E}}_3(z)\, e^{-i\omega t}}\right]$ (from (a))

$$\tilde{\vec{E}}_3(z,t) = \hat{x}\, 2E_0 \cos(kz)\, e^{-i\omega t}$$

Now use Faraday:

$$\vec{\nabla} \times \tilde{\vec{E}}_3 = -\underbrace{\partial \tilde{\vec{B}}_3/\partial t}_{= +i\omega \tilde{\vec{B}}_3(z,t)}$$

$$= -\hat{y}\, 2kE_0 \sin(kz)\, e^{-i\omega t}$$

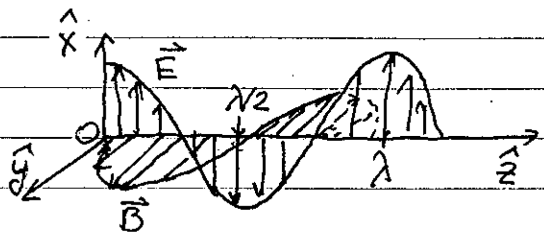
$$\Rightarrow \tilde{\vec{B}}_3(z,t) = \hat{y}\, 2i \underbrace{\frac{k}{\omega}}_{=\frac{1}{c}} E_0 \sin(kz)\, e^{-i\omega t}$$

$$\vec{B}_3(z,t) = \mathrm{Re}[\quad]$$

$$= \hat{y}\, 2\frac{E_0}{c} \sin(kz)\, \mathrm{Re}\left[\overbrace{i e^{-i\omega t}}^{\sin \omega t}\right]$$

$$\boxed{\vec{B}_3(z,t) = \hat{y}\, 2\frac{E_0}{c} \sin(kz) \sin(\omega t)}$$

Result tells us that the standing waves have $\vec{E}$ + $\vec{B}$ out of phase both in space + time.

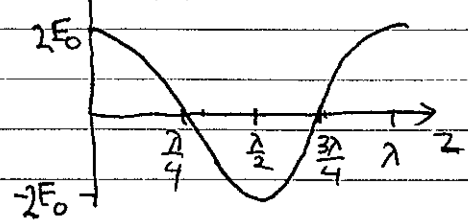


$\vec{E}$ and $\vec{B}$ at $t = T/8$

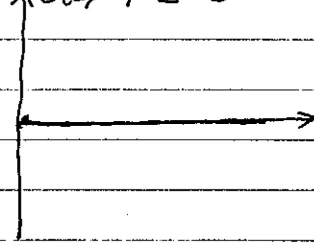
(c)

$t = 0$ and $t = T$

$E(z,0) = 2E_0 \cos kz$

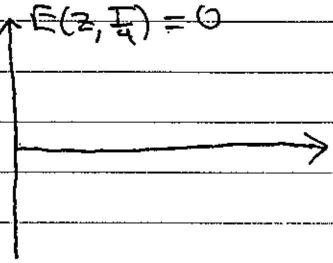


$B(z,0) = 0$

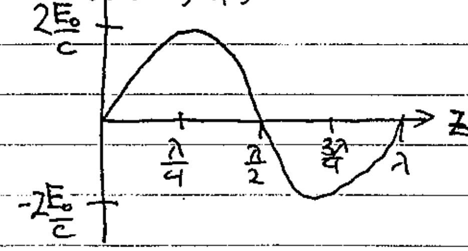


$t = \frac{T}{4}$

$E(z, \frac{T}{4}) = 0$

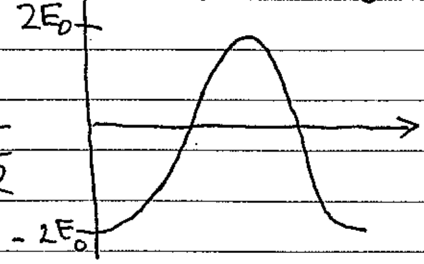


$B(z, \frac{T}{4}) = 2 \sin kz \ \frac{E_0}{c}$

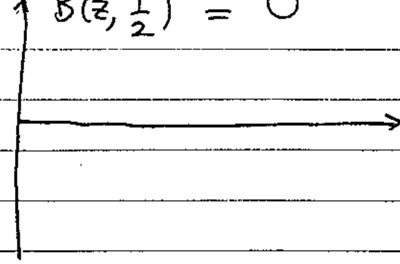


$t = \frac{T}{2}$

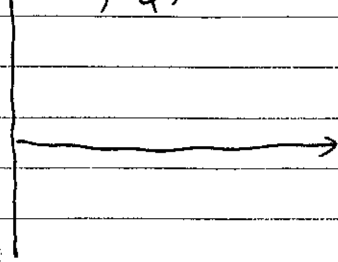
$E(z, \frac{T}{2}) = -2E_0 \cos kz$



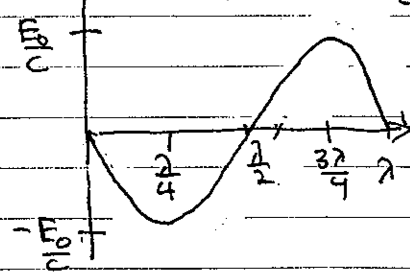
$B(z, \frac{T}{2}) = 0$



$E(z, \frac{3T}{4}) = 0$



$B(z, \frac{3T}{4}) = -2\frac{E_0}{c} \sin kz$



Anti-nodes of E at $m\frac{\lambda}{2}$

nodes of E at $\frac{\lambda}{2}(m+\frac{1}{2})$

$m = 0, \pm 1, \ldots$

Anti-nodes of B at $(m+\frac{1}{2})\frac{\lambda}{2}$

nodes of B at $m\frac{\lambda}{2}$

d) Energy densities

$$u = u_E + u_B$$

$$= \frac{\epsilon_0}{2} \cdot 4E_0^2 \cos^2(kz)\cos^2(\omega t)$$

$$+ \underbrace{\frac{1}{2\mu_0} \cdot 4\frac{E_0^2}{c^2}}_{= \frac{\epsilon_0}{2} \cdot 4E_0^2} \sin^2(kz)\sin^2(\omega t)$$

Time-avg:

$$\langle u_E \rangle = 2\epsilon_0 E_0^2 \cos^2(kz) \langle \underbrace{\cos^2(\omega t)}_{1/2} \rangle$$

$$\boxed{\langle u_E \rangle = \epsilon_0 E_0^2 \cos^2(kz)}$$

$$\boxed{\langle u_B \rangle = \epsilon_0 E_0^2 \sin^2(kz)}$$

$$\langle u \rangle = \langle u_E \rangle + \langle u_B \rangle$$

$$= \epsilon_0 E_0^2 (\cos^2(kz) + \sin^2(kz))$$

$$\boxed{\langle u \rangle = \epsilon_0 E_0^2}$$

e) What is time avg of ~~intensity~~ energy flux, or ~~I~~ I?

$$I = \langle |\vec{S}| \rangle = \frac{1}{\mu_0} \langle |\vec{E} \times \vec{B}| \rangle$$

$$= \frac{1}{\mu_0} \frac{E_0^2}{c} \sin(kz)\cos(kz) \langle \underbrace{\sin(\omega t)\cos(\omega t)}_{= 0, \text{ odd function}} \rangle$$

$$= 0 \,!$$

Standing waves ~~carry~~ transport no energy in either direction → no flux.

4

Formally: $E(x,t) = E_R(x-vt) + E_L(x+vt)$

a) Then: $\boxed{E(x,0) = E_R(x) + E_L(x)}$

and $\frac{\partial E}{\partial t} = -v E_R'(x-vt) + v E_L'(x+vt)$

where the "$'$" means $\frac{d}{du}$ where $u = x - vt$ or $u = x + vt$

Then $\left.\frac{\partial E}{\partial t}\right|_{(x,0)} = -v E_R'(x) + v E_L'(x)$

$$= -v\left(E_R'(x) - E_L'(x)\right)$$

Integrating this equation from $x=0 \to x$:

$$\boxed{\int_0^x \left(\frac{\partial E}{\partial t}\right)\Big|_{(x',0)} dx' = -v\left(E_R(x) - E_L(x)\right)} + v\underbrace{\left(E_R(0) - E_L(0)\right)}_{=0}$$

Using condition $E_R(0) = E_L(0)$

So, solving the 2 boxed eqns for $E_R(x)$ & $E_L(x)$

$$\boxed{\begin{aligned} E_R(x) &= \frac{1}{2}\left(E(x,0) - \frac{1}{v}\int_0^x dx' \left.\frac{\partial E}{\partial t}\right|_{(x',0)}\right) \\ E_L(x) &= \frac{1}{2}\left(\quad '' \quad + \quad '' \quad \right) \end{aligned}}$$

b) Recalling that $E(x,t) = E_R(x-vt) + E_L(x+vt)$

we take and sub $x \to x-vt$ for $E_R(x)$ and

$x \to x+vt$ for $E_L(x)$

$$\Rightarrow E_R(x-vt) = \frac{1}{2}\left(E(x-vt,0) - \frac{1}{v}\int_0^{x-vt} dx' \left.\frac{\partial E}{\partial t}\right|_{(x',0)}\right)$$

$$+ \frac{1}{2}\left(E(x+vt,0) + \frac{1}{v}\int_0^{x+vt} dx' \left.\frac{\partial E}{\partial t}\right|_{(x',0)}\right)$$

Noting that $-\frac{1}{v}\int_0^{x-vt} dx' \left.\frac{\partial E}{\partial t}\right|_{(x',0)} = \frac{1}{v}\int_{x-vt}^{0} dx' \left.\frac{\partial E}{\partial x}\right|_{(x',0)}$

we get:

$$\boxed{E(x,t) = \frac{1}{2}\left[E(x-vt) + E(x+vt)\right] + \frac{1}{v}\int_{x-vt}^{x+vt} dx' \left.\frac{\partial E}{\partial x}\right|_{(x',0)}}$$

c) Since $\vec{B} = B(x,t)\hat{z}$ and $\vec{E} = E(x,t)\hat{y}$

$\rightarrow \quad \vec{\nabla} \times \vec{B} = -\frac{\partial B_z}{\partial x}\hat{y} = -\frac{\partial B}{\partial x}\hat{y}$

and $\frac{1}{v^2}\frac{\partial \vec{E}}{\partial t} = \frac{1}{v^2}\frac{\partial E}{\partial t}\hat{y}$

So $\vec{\nabla}\times\vec{B} = \frac{1}{v^2}\frac{\partial \vec{E}}{\partial t}$ becomes $\frac{\partial B}{\partial x} = -\frac{1}{v^2}\frac{\partial E}{\partial t}$

~~From~~ For part b) we need:

$$\frac{1}{v}\int_{x-vt}^{x+vt} \frac{\partial E}{\partial t}\bigg|_{(x',0)} dx' = -v\int_{x-vt}^{x+vt} dx' \frac{\partial B}{\partial x}\bigg|_{(x',0)}$$

$$= -v\left[B(x+vt,0) - B(x-vt,0)\right]$$

$$\Rightarrow \boxed{E(x,t) = \frac{1}{2}\Big[E(x-vt,0) + vB(x-vt,0) + E(x+vt,0) - vB(x+vt,0)\Big]}$$

d) Integrate above $\frac{\partial}{\partial x}(vB) = -\frac{1}{v}\frac{\partial E}{\partial t}$

R.H.S. from

$$-\frac{1}{v}\frac{\partial E}{\partial t} = -\frac{1}{2v}\Big[-vE'(x-vt,0) - v^2B'(x-vt,0) + vE'(x+vt,0) - v^2B'(x+vt,0)\Big]$$

Integrating both sides w.r.t x:

$$\boxed{vB(x,t) = \frac{1}{2}\Big[E(x-vt,0) + vB(x-vt,0) - E(x+vt,0) + vB(x+vt,0)\Big] + f(t)}$$

integration const.

To find $f(t)$ we use the fact that $B(x,t)$ must obey the wave eqn:

$$\frac{\partial^2 B}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 B}{\partial t^2} = 0$$

$\Rightarrow$ $f(t) = A + Ct$ where A, C are constants

to find A, C use:

$$vB(x,0) = \frac{1}{2}\left[\cancel{E(x,0)} + vB(x,0) - \cancel{E(x,0)} + vB(x,0) + \underbrace{f(t)\big|_{t=0}}_{=A}\right]$$

$$= \cancel{\tfrac{1}{2}}\, vB(x,0) + A$$

$\Rightarrow$ $\boxed{A = 0}$

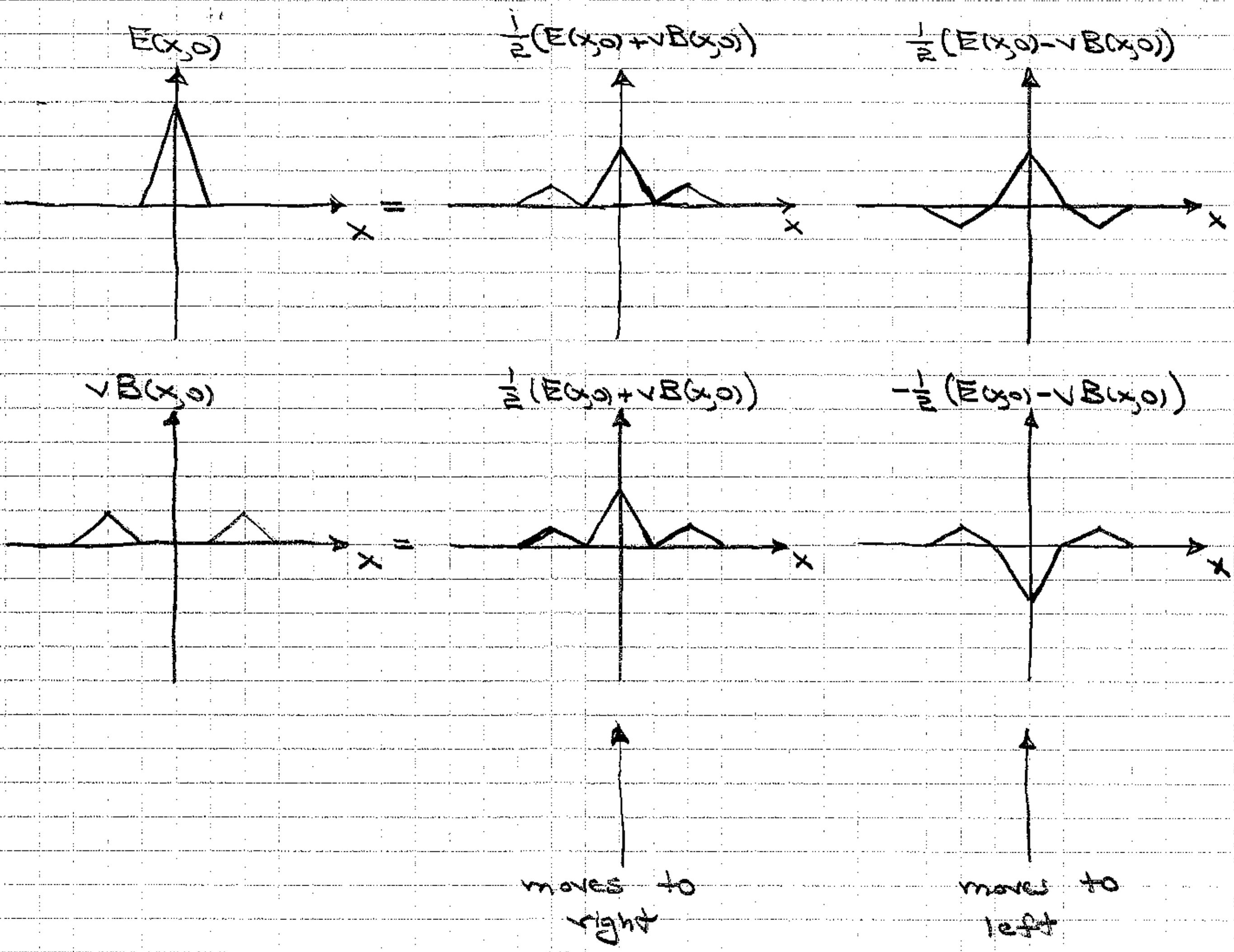
and to find C we use Faraday: $\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$

Doing this you find that $C = 0$ too!

Finally:

$$\boxed{vB(x,t) = \frac{1}{2}\left[E(x-vt,0) + vB(x-vt,0) - E(x+vt,0) + vB(x+vt,0)\right]}$$

(e) $\vec{E} = E(x,t)\,\hat{e}_y$, $\vec{B} = B(x,t)\,\hat{e}_z$



Time $t > 0$: