Homework #5 *due date is Thursday 10/2/2014* ***in class***
(see syllabus about late HWs)
(Note that all Griffiths Problems are from 4^{th} Ed.)

1) A plane-wave solution to Maxwell's equations in a linear, homogeneous dielectric is given by:

$$\mathbf{E}(z,t) = \mathbf{E}_0 \cos\Big(\sqrt{6}z - (6\times 10^{10})t\Big) \ ,$$

where t is in seconds, z is in centimeters, and $\mathbf{E}_0$ is a constant vector.

(a) What is the frequency of the wave? In what part of the electromagnetic spectrum is the wave (radio, microwave, x-ray, etc.)?

(b) What is the index of refraction of the medium?

(c) What is the wavelength of the wave, and what would be the wavelength if the wave emerged from the medium and traveled in free space?

(d) *Give* an expression for the wave's magnetic field.

2) A problem on linear & circular polarization. A monochromatic plane wave propagating in vacuum in the +x direction has electric and magnetic fields given by:

$$\mathbf{E}(x,t) = \mathrm{Re}\big(\tilde{\mathbf{E}}_0 e^{i(kx-\omega t)}\big) \ ,$$
$$\mathbf{B}(x,t) = \mathrm{Re}\big(\tilde{\mathbf{B}}_0 e^{i(kx-\omega t)}\big) \ ,$$

where the vector complex amplitudes (vector phasors) $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are transverse to the propagation direction—i.e., have only y and z components—and are related by

$$\tilde{\mathbf{B}}_0 = \frac{1}{c}\hat{\mathbf{e}}_x \times \tilde{\mathbf{E}}_0 \ .$$

(a) *Linear polarization.* Suppose

$$\tilde{\mathbf{E}}_0 = E_0 \hat{\mathbf{n}} \ ,$$

where E_0 is real and $\hat{\mathbf{n}}$ is a unit vector orthogonal to $\mathbf{e}_x$. Such a wave is said to be *linearly polarized* along $\hat{\mathbf{n}}$. *Find* $\mathbf{E}$ and $\mathbf{B}$ for such a linearly polarized wave. *Determine* and then *describe* or *sketch* the time dependences of the electric and magnetic fields at $x = 0$.

(b) *Find* the instantaneous and average energy density and the instantaneous and average Poynting vector associated with a linearly polarized wave.

(c) *Circular polarization.* Suppose

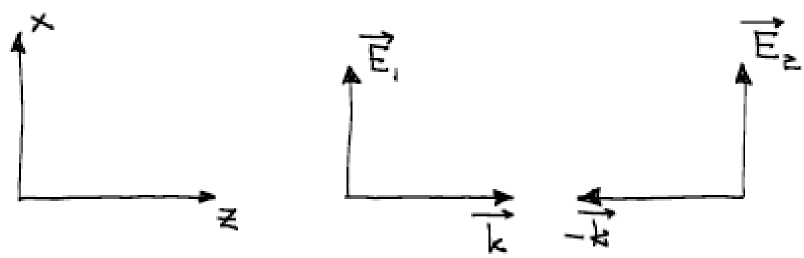
$$\tilde{\mathbf{E}}_0 = E_0(\hat{\mathbf{e}}_y \pm i\hat{\mathbf{e}}_z) \ ,$$

where E_0 is real. Such a wave is called *circularly polarized*; the upper sign corresponds to *right circular polarization* and the lower sign to *left circular polarization*. A circularly polarized wave is an equal linear combination of linearly polarized waves along $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$; the i in the linear combination introduces a 90° phase lag. *Find* $\mathbf{E}$ and $\mathbf{B}$ for such a circularly polarized wave. *Determine* and then *describe* or *sketch* the time dependences of the electric and magnetic fields at $x = 0$ for both left right and left circular polarization.

(d) *Find* the instantaneous and average energy density and the instantaneous and average Poynting vector associated with right and left circularly polarized waves.

3) Problem on standing E&M waves. Consider the superposition of two oppositely propagating monochromatic plane waves with the same frequency, whose electric fields have the same linear polarization and magnitude:

$$\mathbf{E}_1 = \hat{\mathbf{e}}_x E_0 \cos(kz - \omega t) \quad \text{and} \quad \mathbf{E}_2 = \hat{\mathbf{e}}_x E_0 \cos(kz + \omega t) \, .$$



(a) What is the total electric field $\mathbf{E}_3 = \mathbf{E}_1 + \mathbf{E}_2$?

(b) What is the total magnetic field $\mathbf{B}_3$?

This is a standing wave because the nodes of $\mathbf{E}$ and $\mathbf{B}$ don't move. In a traveling wave, the nodes move in the direction of propagation.

(c) *Sketch* $\mathbf{E}_3(z,t)$ and $\mathbf{B}_3(z,t)$ (you should be sketching the single vector component of each) as a function of z over one wavelength at the following times: $t = 0, T/4, T/2, 3T/4, T$, where $T = 2\pi/\omega$ is the period of oscillation.

(d) What are the instantaneous electric and magnetic energy densities as a function of z and t?

(e) What is the time-averaged energy flux (intensity)? *Explain* your answer.

4) A linearly polarized plane wave propagating along the x axis in a material with index of refraction n has electric and magnetic fields

$$\mathbf{E} = E(x,t)\hat{\mathbf{e}}_y \qquad \text{and} \qquad \mathbf{B} = B(x,t)\hat{\mathbf{e}}_z \ .$$

The objective of this problem is to see how the electric and magnetic fields are determined by their values at $t = 0$.

We concentrate first on the electric field. The scalar function $E(x,t)$ satisfies the two-dimensional scalar wave equation, i.e.,

$$\frac{\partial^2 E(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 E(x,t)}{\partial t^2} = 0 \ .$$

The propagation velocity is related to the speed of light in vacuum by $v = c/n$.

The general solution for $E(x,t)$ is

$$E(x,t) = E_R(x - vt) + E_L(x + vt) \ ,$$

where E_R describes a wave propagating in the $+x$ direction and E_L describes a wave propagating in the $-x$ direction. Notice that we have the freedom to add a constant to E_R, while subtracting the same constant from E_L, without affecting the overall solution $E(x,t)$. We can use this freedom to make the difference $E_R(0) - E_L(0)$ equal to anything we want, and throughout this problem we will choose

$$E_R(0) - E_L(0) = 0 \ .$$

The physical reason we have this freedom is that we can't tell which direction a constant "wave" is propagating, so we are free to shuffle the constant back and forth between the right- and left-going waves.

(a) *Find* $E_R(x)$ *and* $E_L(x)$ *in terms of the* $t = 0$ *value of the electric field,* $E(x,0)$*, and the* $t = 0$ *value of the first time derivative of the electric field,*

$$\left.\frac{\partial E(x,t)}{\partial t}\right|_{(x,0)} \ .$$

(b) *Use the result of part (a) to put* $E(x,t)$ *in the form*

$$E(x,t) = \frac{1}{2}\left(E(x - vt, 0) + E(x + vt, 0) + \frac{1}{v}\int_{x-vt}^{x+vt} dx' \left.\frac{\partial E}{\partial t}\right|_{(x',0)}\right) \ .$$

(c) *Use* the Maxwell equation

$$\nabla \times \mathbf{B} = \frac{1}{v^2} \frac{\partial \mathbf{E}}{\partial t}$$

to transform the result of part (b) into the form

$$E(x,t) = \frac{1}{2}\Big(E(x - vt, 0) + vB(x - vt, 0) + E(x + vt, 0) - vB(x + vt, 0)\Big) \, .$$

(d) *Integrate* the same Maxwell equation to show that the magnetic field is given in terms of $t = 0$ quantities by

$$vB(x,t) = \frac{1}{2}\Big(E(x - vt, 0) + vB(x - vt, 0) - E(x + vt, 0) + vB(x + vt, 0)\Big) \, .$$

(e) Suppose the initial electric and magnetic fields are as shown below. *Sketch* the electric and magnetic fields at a time when the right- and left-going waves have moved away from the initial field region. Be sure that your sketch is consistent with the results of parts (c) and (d).