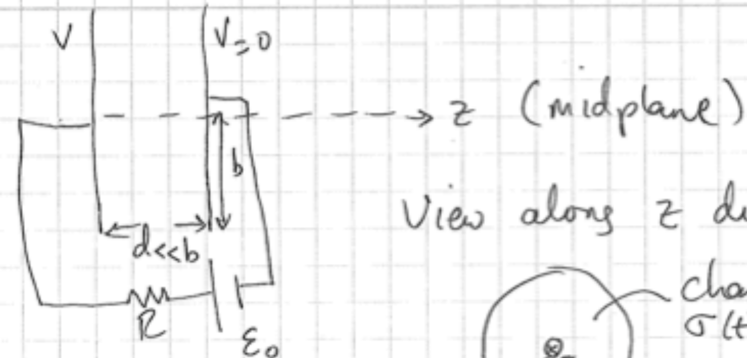
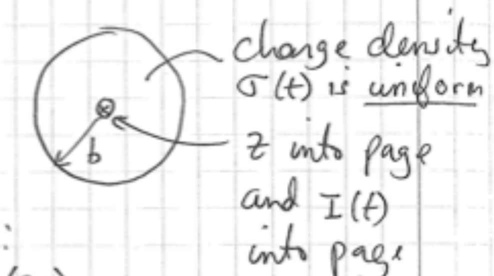
Homework #4 *solution*

1)



View along z direction



At $t=0$ you start charging up the capacitor:

$$Q(t) = C\,\mathcal{E}_0\left(1-e^{-t/RC}\right)$$

$$I(t) = \frac{dQ}{dt} = \frac{\mathcal{E}_0}{R}\,e^{-t/RC}$$

a) In this view:

In time Δt an amount $\Delta Q = I\,\Delta t$ enters the disk and spreads uniformly across the disk. The flow onto the disk is radial as shown. The fraction of ΔQ in outer $>r$ region is just the ratio a areas:

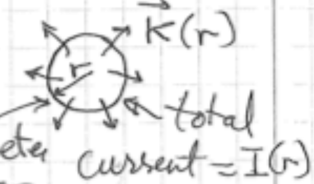
$$\frac{\Delta Q(>r)}{\Delta Q} = \frac{\pi b^2 - \pi r^2}{\pi b^2} = 1 - \frac{r^2}{b^2}$$

Similarly, the current crossing a circle of radius r relative to the total current (at that time t) is

$$I(r)/I = 1 - r^2/b^2$$

The surface current density $\vec{K}(r)$ is just

$$\boxed{\vec{K}(r) = \frac{I(r)}{2\pi r}\hat{r}}$$



$$\boxed{\vec{K}(r,t) = \frac{I(t)}{2\pi r}\left(1 - r^2/b^2\right)\hat{r}}$$

total current entering disk at time t.

b) Displacement current $\vec{J}_D = \epsilon_0 \partial \vec{E}/\partial t$

Between plates $\vec{E}$ is uniform (at a fixed t)

$$\vec{E}(t) = \frac{\sigma(t)}{\epsilon_0}\hat{z} = \frac{Q(t)}{\epsilon_0 \pi b^2}\hat{z}$$

$$d\vec{E}/dt = \frac{I(t)}{\epsilon_0 \pi b^2}\hat{z}$$

$$\rightarrow \boxed{\vec{J}_D = \frac{I(t)}{\pi b^2}\hat{z}}$$

Total displacement current is $I_D(t) = \int_S \vec{J}\cdot d\vec{a}$

$I_D = I(t)$ as expected

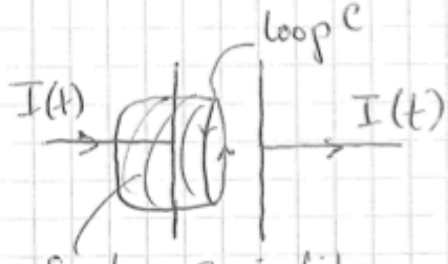
c) $\vec{B}(r,t)$ between plates. Use Maxwell-Ampere:

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \underbrace{I_{enc}}_{=0} + \mu_0 \epsilon_0 \int_S \vec{J}_D \cdot da$$

$$B\, 2\pi r = \frac{\mu_0}{\pi b^2} I(t) \cdot \pi r^2 = \mu_0 I(t) r^2/b^2$$

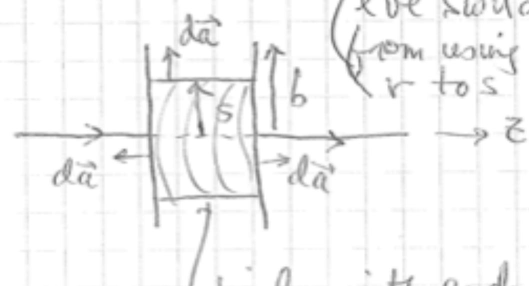
$$\boxed{\vec{B}(r,t) = \mu_0 I(t) r \hat{\phi}/2\pi b^2}$$

Note that you should get the same answer when S encloses $I(t)$:



Surface S is like a soap bubble ~~being~~ that extends to the left with $I(t)$ poking thru it.

d) Here's the picture:



(I've switched from using r to s)

Cylinder with ends up against cap. plates

On the surface of this cylinder $d\vec{a}$ is either radial $(\hat{s})$ or along $-\hat{z}$ or $+\hat{z}$ at the ends. The $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$ has only radial components:

Between plates: $\begin{cases} \vec{E}(t) = \dfrac{Q(t)}{\epsilon_0 \pi b^2}\hat{z}\ ; \ s \le b \\ \vec{B}(s,t) = \dfrac{\mu_0 I(t) s}{2\pi b^2}\hat{\phi} \end{cases}$

$$\frac{1}{\mu_0}\vec{E}\times\vec{B} = \vec{S} = \frac{Q(t)I(t)s}{\epsilon_0 2\pi^2 b^4}(-\hat{s})$$

$\rightarrow$ Power transmitted thru this cylinder is just

$$P = \oint_S \vec{S}\cdot d\vec{a}$$

$$P = \oint_S \vec{S}\cdot \underbrace{d\vec{a}}_{-da\,\hat{s}} = + \frac{Q(t)I(t)}{\epsilon_0 2\pi^2 b^4} s \int_0^d dz \underbrace{\int_0^{2\pi} s\, d\phi}_{2\pi s d}$$

non-zero only from radial side of cyl.

~~power thru cylinder inward~~

$$\boxed{P = \frac{Q(t)\, I(t)\, s^2 d}{\epsilon_0 \pi b^4} = \frac{Q(t)\, I(t)}{C}\left(\frac{s}{b}\right)^2}$$

Power transmitted into cylinder $\rightarrow d\vec{a} = -da\,\hat{s}$

Using capacitance $C = \epsilon_0 \pi b^2/d$

Power is energy per time. Total energy U transported ~~out~~ inward thru cylinder is just

$$U = \int_0^\infty P\, dt = \frac{1}{C}\left(\frac{s}{b}\right)^2 \int_0^\infty Q(t)\, I(t)\, dt$$

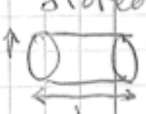
$$= \int_0^\infty C\, \mathcal{E}_0 (1 - e^{-t/RC}) \frac{\mathcal{E}_0}{R} e^{-t/RC}\, dt$$

$$= \frac{C\, \mathcal{E}_0^2}{R} \underbrace{\left[-RC\, e^{-t/RC} + \frac{RC}{2} e^{-2t/RC} \right]_0^\infty}_{+\frac{RC}{2}}$$

$$= C^2 \mathcal{E}_0^2 / 2$$

Electrostatic Energy stored in this volume between plates $\rightarrow$

$$\boxed{U = \frac{1}{2} C\, \mathcal{E}_0^2 \left(\frac{s}{b}\right)^2}$$



Note that this is the fraction of energy stored within the volume bounded by s in capacitor. Total is when $s = b$, $U = \frac{1}{2} C \mathcal{E}_0^2$

2)

a. We have simply

$$\mathcal{P}_{\rm em} \equiv \epsilon_0 \vec{E} \times \vec{B} = \epsilon_0 EB\, \hat{z} \times \hat{x} = \epsilon_0 EB\hat{y}$$

$$\Longrightarrow \vec{p}_{\rm em} = Ad\mathcal{P}_{\rm em} = \epsilon_0 EBAd\hat{y}\ .$$

b. We know that a straight current moving in the neighborhood of a uniform magnetic field will feel a force

$$\vec{F} = I\,\vec{\ell} \times \vec{B} = IdB\,\hat{z} \times \hat{x} = IBd\,\hat{y} = Bd\left(-\frac{dQ}{dt}\right)\hat{y}\ ,$$

where the minus sign is because the charge is decreasing over time.

Now, to determine the total impulse, i.e., the total change in momentum of the wire as a result of this force acting on it, while the total charge decreases, we must integrate over time:

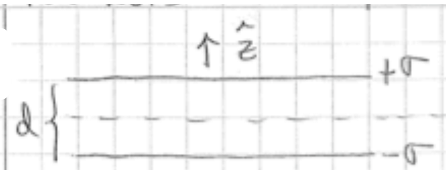
$$\Delta\vec{p} = \int dt\, \vec{F} = -\int_0^\infty dt \frac{dQ}{dt} Bd\hat{y} = -[Q(\infty) - Q(0)]Bd\hat{y} = QBd\hat{y}\ .$$

However, we also know that $Q = \sigma A = \epsilon_0 EA$, so that this may be written as

$$\Delta\vec{p} = \epsilon_0 EABd\hat{y}\ ,$$

which is the same as the momentum between the plates before the wire was inserted.

3)



Infinite plane cap.

a) $\overleftrightarrow{T} = ?$ $\quad T_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \delta_{ij}\left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right)$

For us $\vec{E} = -\frac{\sigma}{\epsilon_0}\hat{z}$

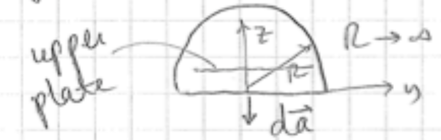
between plates, zero outside. And $\vec{B} = 0$ everywhere

So: $\quad T_{ij} = \epsilon_0 E_i E_j - \delta_{ij} \frac{\epsilon_0 E^2}{2}$

$$\boxed{T_{xx} = -\frac{\epsilon_0 E^2}{2} = T_{yy} \;,\; T_{zz} = +\frac{\epsilon_0 E^2}{2}}$$

all others $= 0$

Calculate force on upper plate. Use hemisphere of ∞ radius with x-y plane as midplane:



Hemisphere surface contribute nothing to the integral (no $\vec{E}$)

Force:

$$\vec{F} = \oint_S \overleftrightarrow{T} \cdot d\vec{a} = \int T_{zz}\, da_z (-\hat{z}) = -T_{zz} A \hat{z}$$

$$\boxed{\vec{f} = \frac{\vec{F}}{A} = -\frac{\epsilon_0 E^2}{2}\hat{z} = -\frac{\sigma^2}{2\epsilon_0}\hat{z}}$$

b) Momentum per unit area per unit time crossing midplane from upper plate: $\Delta\vec{P}/\Delta t = \int_S (-\overleftrightarrow{T}) \cdot d\vec{a}$

Here, $-\overleftrightarrow{T}$ is the momentum flux density across $d\vec{a}$. So:

$$\frac{\Delta\vec{p}}{\Delta t} = \int_S (-\overleftrightarrow{T}) \cdot \overbrace{d\vec{a}}^{-da\,\hat{z}} = -T_{zz} \cdot (-\hat{z}) A$$

$$\boxed{\frac{\Delta\vec{p}}{A\,\Delta t} = +\frac{\epsilon_0 E^2}{2}\hat{z}}$$

The momentum flux ~~through the midplane~~ from upper plate out through the midplane is in the $+\hat{z}$ direction.

c) The ~~change~~ momentum per time per area that left the upper plate (calculated in b)) gives rise to ~~the~~ its "recoil". Which direction does it recoil? Well, it (the upper plate) had some momentum, $\vec{p}_i$, then it lost some mom. $\Delta\vec{p}$ (part b) and ended up with $\vec{p}_f = \vec{p}_i - \Delta\vec{p}$

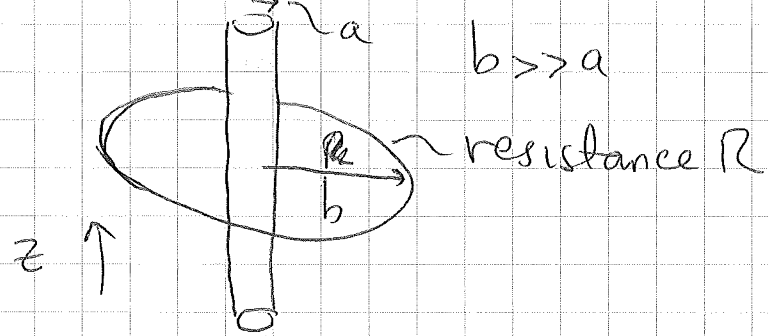
$$\Rightarrow \boxed{\vec{f}_{recoil} = \frac{\vec{p}_f - \vec{p}_i}{A\,\Delta t} = -\frac{\Delta\vec{p}}{A\,\Delta t} = -\frac{\epsilon_0 E^2}{2}\hat{z}}$$

Makes sense! The momentum lost by upper plate is along $+\hat{z}$ direction resulting in a "recoil" in the $-\hat{z}$ direction.

4) A long thin solenoid, carrying current I, has n turns / length. Coaxial with the solenoid is a large circular ring of wire with resistance R:

inside solenoid: $\vec{B} = \mu_0 n I \hat{z}$ (Ampere)

outside $\vec{B} = 0$



$b \gg a$

a) Gradually decrease I in solenoid. This will induce a current I_i in the ring.

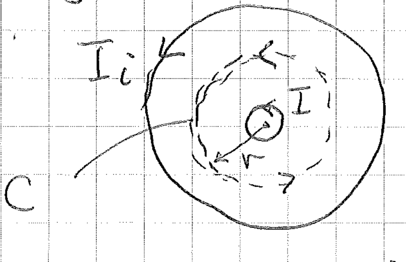
$$\underbrace{\mathcal{E}}_{\text{induced in ring}} = -d\Phi/dt = -\mu_0 n \frac{d}{dt} \underbrace{\int I \, da}_{\pi a^2}$$

$$= -\mu_0 n \frac{dI}{dt} \pi a^2$$

$$I_i = \mathcal{E}/R = -\frac{\mu_0 n \pi a^2}{R} \frac{dI}{dt}$$

The direction of this current will be in <u>same</u> direction as I in the solenoid, by Lenz's law. Also note that the $-$ sign is there because dI/dt is negative, and this gives I_i as $+$. If dI/dt was $+$, we'd get a negative I_i telling us that <u>its</u> direction is opposite of I.

b) Looking down on solenoid / ring:



$\vec{B}$ and $\hat{z}$ out of page with I counter-clockwise.

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

Here we want the $\vec{E}$ and $\vec{B}$ just outside the solenoid. The $\vec{B}$ has to be that due to the induced current, I_i, in the ring. The $\vec{E}$ ~~due to~~ is from Faraday

$$\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$$

$$\underbrace{\oint_C \vec{E}\cdot d\vec{l}}_{E\,2\pi r} = -\frac{d\Phi}{dt} = \mathcal{E} = -\mu_0 n\pi a^2\frac{dI}{dt}$$

$$\boxed{\vec{E} = -\frac{\mu_0 n a^2}{2r}\frac{dI}{dt}\hat{\phi}}$$

negative

(this is also c.c.w. above in figure)

$\vec{B}$ on the axis of a circular ring is:

$$\boxed{\vec{B} = \frac{1}{2}\mu_0 I_i\frac{b^2\,\hat{z}}{(b^2+z^2)^{3/2}}}$$

We can make the assumption that this $\vec{B}$ is approximately the $\vec{B}$ on the outer surface of the solenoid because of $a \ll b$. So

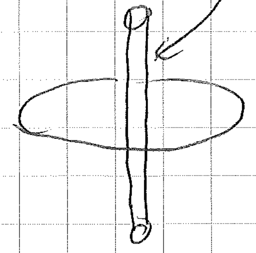
$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}\Big|_{r=a} = -\frac{n a \mu_0 I_i}{4}\frac{dI}{dt}\frac{b^2\,\hat{r}}{(b^2+z^2)^{3/2}}$$

Points away from solenoid towards ring $\left(\frac{dI}{dt}<0\right)$

The power is found by $P = \int_S \vec{S}\cdot d\vec{a}$

Here the surface S is the outer surface of the solenoid.

S



↳ $d\vec{a} = \hat{r}\, 2\pi a\, dz$

We can take the dz integral from $-\infty \to \infty$ because solenoid is long

$\vec{B}$

and the field of the ring falls as $\frac{1}{z^3}$ for large z. So:

$$P = \int_S \vec{S}\cdot d\vec{a} = -\frac{na}{4}\mu_0 I_i \frac{dI}{dt}\, b^2 \underbrace{\int_{-\infty}^{\infty}\frac{2\pi a\, dz}{(b^2+z^2)^{3/2}}}_{2\pi a\cdot 2/b^2}$$

$$P = -\pi a^2 \mu_0 n I_i \frac{dI}{dt}$$

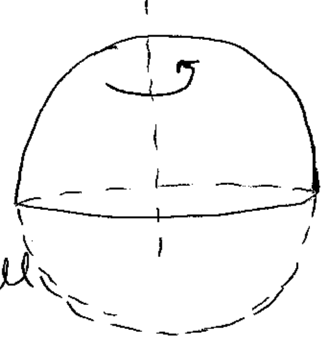
sub $\frac{dI}{dt} = -\frac{I_i R}{\mu_0 n \pi a^2}$ (part a)

→ $\boxed{P = I_i^2 R}$ ← This came from Poynting and is <u>consistent</u> with the energy dissipated in the ring from a current I_i passing thru a resistance $R \Rightarrow I_i^2 R$!

Prob 8.3, Griffiths

From Example 5.11,

$$\vec{A}(\vec{r}) = \frac{\mu_0 R^4 \sigma}{3 r^3}\, \vec{\omega} \times \vec{r}$$



for points outside the shell

$$\Rightarrow \quad \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) = \frac{\mu_0 R^4 \sigma}{3}\, \vec{\nabla} \times \left(\vec{\omega} \times \frac{\vec{r}}{r^3}\right)$$

But $\dfrac{\vec{r}}{r^3} = -\vec{\nabla}\left(\dfrac{1}{r}\right) \Rightarrow \vec{B}(\vec{r}) = -\dfrac{\mu_0 R^4 \sigma}{3}\, \underbrace{\vec{\nabla} \times \left(\vec{\omega} \times \vec{\nabla} \frac{1}{r}\right)}_{\left[\vec{\omega}\, \vec{\nabla} \cdot \vec{\nabla}(1/r) - (\vec{\omega} \cdot \vec{\nabla}) \vec{\nabla}(1/r)\right]}$

But $\nabla^2\left(\frac{1}{r}\right) = 0$ for $r \neq 0$

$$\Rightarrow \quad \vec{B}(\vec{r}) = \frac{\mu_0 R^4 \sigma \omega}{3} \frac{\partial}{\partial z} \vec{\nabla}\left(\frac{1}{r}\right) \text{ for } r \geq R$$

But $\vec{B}$ = uniform $= 2\dfrac{\mu_0 R \sigma}{3} \vec{\omega}$ for $r < R$

The net (mag.) force on the upper hemisphere is given by integrating the magnetic part of the Maxwell stress tensor over a closed surface that encloses that hemisphere. Let us take that surface to be outside but infintesimally close to the hemisphere and its base. Since the force is only in the $\hat{z}$ direction

$$\left(\int \overleftrightarrow{T}_{mag} \cdot d\vec{a}\right)_z = \int T^{(mag)}_{zx}\, da_x + T^{(mag)}_{zy}\, da_y + T^{(mag)}_{zz}\, da_z$$

On the "bowl",

$$T^{(mag)}_{zx} = \frac{1}{\mu_0} B_z B_x ; \quad T^{(mag)}_{zy} = \frac{1}{\mu_0} B_z B_y ; \quad T^{(mag)}_{zz} = \frac{1}{\mu_0}\left(B_z^2 - \frac{1}{2}B^2\right) = \frac{1}{\mu_0}\left(\frac{1}{2}B_z^2 - \frac{1}{2}B_x^2 - \frac{1}{2}B_y^2\right)$$

with
$$B_z = \frac{\mu_0 R^4 \sigma\omega}{3} \frac{\partial^2}{\partial z^2}\frac{1}{r} = \frac{\mu_0 R^4\sigma\omega}{3}\frac{\partial}{\partial z}\left(-\frac{z}{r^3}\right) = \frac{\mu_0 R^4 \sigma\omega}{3}\left(\frac{3z^2 - r^2}{r^5}\right);$$

$$B_x = \frac{\mu_0 R^4\sigma\omega}{3}\frac{\partial}{\partial x}\left(-\frac{z}{r^3}\right) = \frac{\mu_0 R^4 \sigma\omega}{\cancel{3}} \cdot \frac{\cancel{3}zx}{r^5} = \frac{\mu_0 R^4\sigma\omega\, zx}{r^5};$$

$$B_y = \frac{\mu_0 R^4 \sigma\omega}{r^5} zy$$

Thus,
$$\left(\int_{Bowl} \overleftrightarrow{T}_{mag} \cdot d\vec{a}\right)_z = \mu_0 R^2 \frac{R^8 \sigma^2\omega^2}{3} \int_0^{2\pi}\!\!\int_0^{\pi/2} \frac{(3\cos^2\theta - 1)\cos\theta \sin\theta\cos\phi}{R^6} \times \sin\theta\cos\phi \cdot \sin\theta\, d\theta\, d\phi$$

$$+ \mu_0 \frac{R^2 R^8 \sigma^2 \omega^2}{3}\int_0^{2\pi}\!\!\int_0^{\pi/2} \frac{(3\cos^2\theta - 1)\cos\theta\sin\theta\sin\phi}{R^6} \sin\theta\sin\phi \cdot \sin\theta\, d\theta\, d\phi$$

$$+ \mu_0 R^2 R^8 \sigma^2 \omega^2 \int_0^{2\pi}\!\!\int_0^{\pi/2}\left[\frac{1}{2}\frac{1}{9}\frac{(3\cos^2\theta - 1)^2}{R^6} - \frac{1}{2}\frac{(\cos\theta\sin\theta\cos\phi)^2}{R^6} - \frac{1}{2}\frac{(\cos\theta\sin\theta\sin\phi)^2}{R^6}\right] \times (\sin\theta\, d\theta\, d\phi) \times \underbrace{\cos\theta}_{n_z}$$

(since $da_x = (R^2\sin\theta\, d\theta\, d\phi) \times n_x = R^2\sin\theta\, d\theta\, d\phi \times \sin\theta\cos\phi$, etc.)

Use: $\int_0^{2\pi} \sin^2\phi \, d\phi = \int_0^{2\pi} \cos^2\phi \, d\phi = \pi$

$$\int_0^{\pi/2} (3\cos^2\theta - 1)\cos\theta \sin^2\theta \sin\theta \, d\theta = \int_0^1 (3\mu^2-1)\mu(1-\mu^2)\, d\mu$$

$$= \int_0^1 (3\mu^3 - 3\mu^5 - \mu + \mu^3)\, d\mu \qquad \begin{array}{l} \mu = \cos\theta \\ d\mu = -\sin\theta \, d\theta \end{array}$$

$$= \frac{3}{4} - \frac{\cancel{3}}{\cancel{6}_2} - \frac{1}{2} + \frac{1}{4}$$

$$= 0$$

$$\int_0^{\pi/2} \left[\frac{1}{2}(\cos^2\theta - 1/3)^2 - \frac{1}{2}\cos^2\theta \sin^2\theta\right] \sin\theta \, d\theta \cos\theta$$

$$= \frac{1}{2}\int_0^1 \left[(\mu^2 - 1/3)^2 - \mu^2(1-\mu^2)\right]\mu \, d\mu$$

$$= \frac{1}{2}\int_0^1 \mu(\mu^4 - 2/3\,\mu^2 + 1/9 - \mu^2 + \mu^4)\, d\mu$$

$$= \frac{1}{2}\int_0^1 (2\mu^5 - 5/3\,\mu^3 + 1/9\,\mu)\, d\mu = \frac{1}{2}\left(\frac{2}{6} - \frac{5}{3}\cdot\frac{1}{4} + \frac{1}{9}\cdot\frac{1}{2}\right)$$

$$= \frac{1}{2}\left(-\frac{1}{12} + \frac{1}{18}\right) = -\frac{1}{12}\cdot\frac{1}{6} = -\frac{1}{72}$$

Thus, $$\left(\int_{\text{Bowl}} \overleftrightarrow{T}_{mag} \cdot d\vec{a}\right)_z = -\frac{\mu_0 R^4 \sigma^2 \omega^2}{\cancel{72}\,36} \cdot \cancel{2}\pi \leftarrow \text{from the } \phi \text{ integration}$$

On the "equatorial disk", since $d\vec{a}$ is along $-\hat{z}$, and $\vec{B}$ is uniform everywhere and along z,

$$\left(\int_{\text{disk}} \overleftrightarrow{T}_{mag} \cdot d\vec{a}\right)_z = -T_{zz}^{(mag)}(r<R) \cdot \underbrace{\pi R^2}_{\text{disk area}}$$

$$= -\frac{1}{2\mu_0}\left(B_z^2 - B_x^2 - B_y^2\right) \cdot \pi R^2$$

$$= \frac{1}{2} \cdot \frac{4\mu_0 R^2 \sigma^2 \omega^2}{9} \cdot \pi R^2 = -\mu_0 R^4 \sigma^2 \omega^2 \frac{2\pi}{9}$$

Thus, $$\left(\int_{\text{hemisphere}} \overleftrightarrow{T}_{mag} \cdot d\vec{a}\right)_z = -\mu_0 R^4 \sigma^2 \omega^2 \left(\frac{2\pi}{9} + \frac{\pi}{36}\right) = -\mu_0 R^4 \sigma^2 \omega^2 \frac{\pi}{4}, \text{ along } \hat{z} \text{ (attraction)}$$