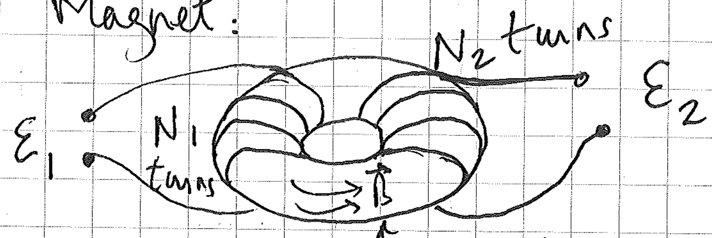
7.57) Transformer has same $\vec{B}$ field going thru 2 separate coils that consist of independant circuits. E.g. 2 coils wrapped around a toroidal magnet:



iron torus with $\vec{B}$ in it due to changing flux induced by one of the circuits (1 or 2)

The main point is that the same flux goes through a loop of either circuit:

$$\Phi = \text{flux/turn} = \text{same for 1 or 2}$$

$$\rightarrow \Phi_1 = N_1 \Phi \quad \text{and} \quad \Phi_2 = N_2 \Phi$$

So;
$$\mathcal{E}_1 = -d\Phi_1/dt = -N_1\, d\Phi/dt$$
$$\mathcal{E}_2 = \quad = -N_2\, d\Phi/dt$$

Ratio of these gives:

$$\boxed{\mathcal{E}_1/\mathcal{E}_2 = N_1/N_2}$$

This provides a mechanism for "stepping up" or "stepping down" the voltage of a circuit using a transformer.

7.58) Energy conservation in transformers.

→ stepping up or down must conserve energy

a) Show that $M^2 = L_1 L_2$ in tformer.

I_1 + I_2 are respective currents and by definition:

$$\Phi_1 = I_1 L_1 + M I_2 = N_1 \Phi$$
$$\Phi_2 = I_2 L_2 + M I_1 = N_2 \Phi$$

$$\Phi = \frac{I_1 L_1 + M I_2}{N_1} = \frac{I_2 L_2 + M I_1}{N_2}$$

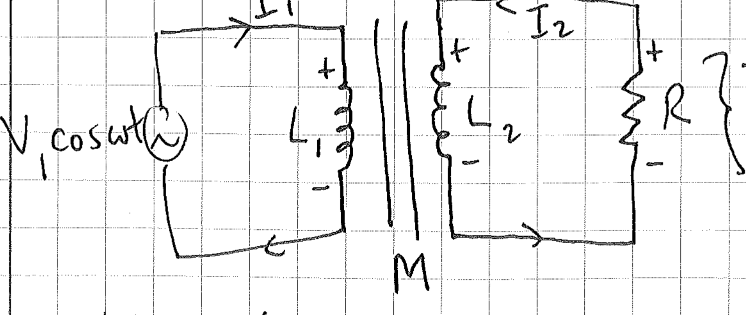
this eqn holds for all I_1, I_2.

Specifically, if $I_1 = 0$, then

$$\left.\begin{aligned} \frac{M \cancel{I_2}}{N_1} &= \frac{\cancel{I_2} L_2}{N_2} \\ \text{and if } I_2 = 0: \quad \frac{L_1}{N_1} &= \frac{M}{N_2} \end{aligned}\right\} \text{divide these eqns: } \frac{M}{L_1} = \frac{L_2}{M}$$

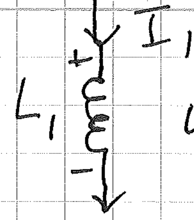
$$\boxed{M^2 = L_1 L_2}$$

b) Can show this using circuit diagram of 2 coils =



$-V = I_2 R$ (I_2 direction choice gives sign)

Kirchoff: $V_1 \cos \omega t = V_{L_1}$ and $V_{L_2} = -I_2 R$

Also from convention 

we have:

$$V_{L_1} = L\frac{dI_1}{dt}$$

Finally we add contribution from mutual induction:

$$① \quad V_{L_1} = L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt} = V_1\cos\omega t$$

$$② \quad V_{L_2} = L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt} = -I_2 R$$

Can also use part a) results and note that voltages in 2 ~~cw~~ circuits will be different signs (right hand rule) due to Lenz's law.

c) Solve equations in part b) for I_1 & I_2

Multiply ① above by L_2 and sub ② in form

$$L_2\frac{dI_2}{dt} = -I_2 R - M\frac{dI_1}{dt} \text{ into it:}$$

$$L_2 V_1 \cos\omega t = \underbrace{L_1 L_2}_{\substack{M^2 \\ \text{from a)}}}\frac{dI_1}{dt} + M\left(-I_2 R - M\frac{dI_1}{dt}\right)$$

$$L_2 V_1 \cos\omega t = M^2\frac{dI_1}{dt} - MI_2 R - M^2\frac{dI_1}{dt}$$

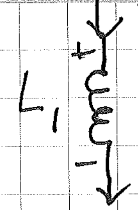
$$= -MI_2 R$$

$$\rightarrow \boxed{I_2 = -\frac{L_2 V_1 \cos\omega t}{MR}}$$

Sub back into ① above

sub back

Also from convention



we have:

$$V_{L_1} = L\frac{dI_1}{dt}$$

Finally we add contribution from mutual induction:

$$\boxed{\begin{aligned} ①\ & V_{L_1} = L_1 \frac{dI_1}{dt} + M\frac{dI_2}{dt} = V_1 \cos\omega t \\ ②\ & V_{L_2} = L_2 \frac{dI_2}{dt} + M\frac{dI_1}{dt} = -I_2 R \end{aligned}}$$

Can also use part a) results and note that Voltages in 2 ~~cw~~ circuits will be different signs (right hand rule) due to Lenz's law.

c) Solve equations in part b) for I_1 & I_2

Multiply ① above by L_2 and sub ② in form $L_2\frac{dI_2}{dt} = -I_2R - M\frac{dI_1}{dt}$ into it:

$$L_2V_1\cos\omega t = \underbrace{L_1L_2}_{M^2}\frac{dI_1}{dt} + M\left(-I_2R - M\frac{dI_1}{dt}\right)$$

from a)

$$L_2V_1\cos\omega t = M^2\frac{dI_1}{dt} - MI_2R - M^2\frac{dI_1}{dt}$$

$$= -MI_2R$$

$$\rightarrow \boxed{I_2 = -\frac{L_2V_1\cos\omega t}{MR}}$$

Sub back into ① above

$$V_1 \cos\omega t = L_1 \frac{dI_1}{dt} + M \underbrace{\frac{dI_2}{dt}}_{\frac{L_2 V_1 \omega}{MR}\sin\omega t}$$

$$\frac{dI_1}{dt} = \frac{V_1}{L_1}\left(\cos\omega t - \frac{L_2}{R}\omega \sin\omega t\right)$$

$$\boxed{I_1 = \frac{V_1}{L_1}\left[\frac{\sin\omega t}{\omega} + \frac{L_2}{R}\cos\omega t\right]}$$

d) Ratio of voltages $V_{out}/V_{in} = ? = \frac{I_2 R}{V_1 \cos\omega t}$

$$= \left(-\frac{L_2 V_1}{MR}\cos\omega t\right)\frac{R}{V_1\cos\omega t}$$

$$= -\frac{L_2}{M} = -\frac{N_2}{N_1}$$

from work done in part a)

e) Calc. input/output powers & show (energy conservation) that their averages are equal:

$$P_{in} = V_{in} I_1 = (V_1 \cos\omega t)\frac{V_1}{L_1}\left[\frac{\sin\omega t}{\omega} + \frac{L_2}{R}\cos\omega t\right]$$

$$\boxed{P_{in} = \frac{V_1^2}{L_1}\left(\frac{\sin\omega t\cos\omega t}{\omega} + \frac{L_2}{R}\cos^2\omega t\right)}$$

Time avg over a full cycle will give Ø for 1st term and $\frac{1}{2}$ for $\cos^2\omega t$ term, so

$$\boxed{\langle P_{in}\rangle = \frac{1}{2}\frac{V_1^2 L_2}{R L_1}}$$

$$P_{out} = V_{out} I_2 = I_2^2 R = \left(\frac{L_2 V_1}{M}\right)^2 \frac{1}{R} \cos^2 \omega t$$

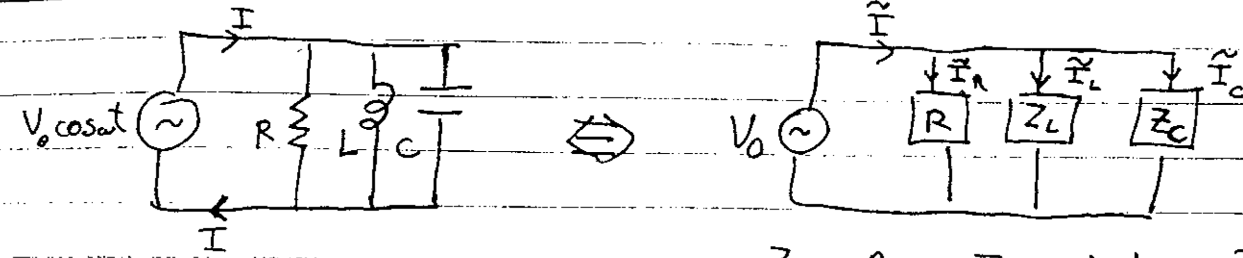
$$(M^2 = L_1 L_2)$$

$$\boxed{\langle P_{out} \rangle = \frac{1}{2} \frac{V_1^2}{R} \frac{L_2}{L_1} = \langle P_{in} \rangle}$$

Physics 406

Problem Set #3 Solutions

Problem 1: Parallel R-L-C circuit



$$Z_R = R \quad Z_L = -i\omega L \quad Z_C = \frac{i}{\omega C}$$

Since the elements are in parallel

$$V_R = V_L = V_C = V_0$$

The total current divides between the elements

$$\tilde{I} = \tilde{I}_R + \tilde{I}_L + \tilde{I}_C = \tilde{V}_R/Z_R + \tilde{V}_C/Z_C + \tilde{V}_L/Z_L$$

$$\Rightarrow \tilde{I} = \left(\frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L}\right) V_0 = \frac{V_0}{Z_{total}}$$

Parallel addition of impedances

$$\frac{1}{Z_{total}} = \frac{1}{Z_R} + \frac{1}{Z_C} + \frac{1}{Z_L} = \frac{1}{R} - i\omega C + \frac{i}{\omega L}$$

(a) On resonance $\omega = \frac{1}{\sqrt{LC}}$

$$\Rightarrow \frac{1}{Z_{total}} = \frac{1}{R} - i\sqrt{\frac{C}{L}} + i\sqrt{\frac{C}{L}} = \frac{1}{R}$$

$$\Rightarrow \tilde{I} = \frac{V_0}{R} \Rightarrow \boxed{I(t) = Re(\tilde{I}e^{-i\omega t}) = \frac{V_0}{R}\cos\omega t}$$

On resonance the magnitude of $|Z_L| = |Z_C| = \sqrt{\frac{L}{C}}$ but they are 180° out of phase so they cancel

On resonance a parallel L-C is like an open circuit.

(b) In general

$$I(t) = \mathrm{Re}\left(\tilde{I}\, e^{-i\omega t}\right) = \mathrm{Re}\left(\frac{V_0}{Z_{total}}\, e^{-i\omega t}\right)$$

$$= \frac{V_0}{|Z_{total}|} \cos\left(\omega t + \mathrm{Arg}(Z_{total})\right)$$

$$\frac{1}{|Z_{total}|} = \left|\frac{1}{Z_{total}}\right| = \left|\frac{1}{R} - i\left(\omega C - \frac{1}{\omega L}\right)\right|$$

$$= \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} = \frac{1}{\omega L}\sqrt{\frac{\omega^2 L^2}{R^2} + (\omega^2 LC - 1)^2}$$

$$= \frac{1}{R}\,\frac{\Gamma}{\omega}\sqrt{\frac{\omega^2}{\Gamma^2} + \left(\frac{\omega^2}{\omega_0^2} - 1\right)}, \quad \Gamma = \frac{R}{L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\mathrm{Arg}(Z_{total}) = -\mathrm{Arg}\left(\frac{1}{Z}\right) = \tan^{-1}\left(\frac{\omega C - \frac{1}{\omega L}}{1/R}\right)$$

$$= \tan^{-1}\left(\frac{\Gamma}{\omega}\left(\frac{\omega^2}{\omega_0^2} - 1\right)\right)$$

$$\therefore \boxed{\begin{aligned} I(t) &= \frac{V_0}{R}\,\frac{\Gamma}{\omega}\sqrt{\frac{\omega^2}{\Gamma^2} + \left(\frac{\omega^2}{\omega_0^2} - 1\right)}\cos\left(\omega t + \mathrm{Arg}(Z_{total})\right) \\ \mathrm{Arg}(Z_T) &= \tan^{-1}\left(\frac{\Gamma}{\omega}\left(\frac{\omega^2}{\omega_0^2} - 1\right)\right)\end{aligned}}$$

Note: On resonance, $I(t) = \frac{V_0}{R}\cos\omega t$

$\mathrm{Arg}(Z_{total}) = 0$

Problem 2 Quality factor

(a) Math Lemma: Given $A(t) = A_0 \cos(\omega t - \phi_A)$
$B(t) = B_0 \sin(\omega t - \phi_B)$

Show $\langle A(t)\, B(t)\rangle \equiv \frac{1}{T}\int_0^T dt\, A(t) B(t) = \frac{1}{2}\mathrm{Re}(\tilde{A}^* \tilde{B})$
$= \frac{1}{2}\mathrm{Re}(\tilde{A}\tilde{B}^*)$

where $\tilde{A} \equiv A_0 e^{i\phi_A}$ $\quad A(t) = \mathrm{Re}(\tilde{A} e^{-i\omega t})$
$\tilde{B} = B_0 e^{i\phi_B}$ $\quad B(t) = \mathrm{Re}(\tilde{B} e^{-i\omega t})$

Proof: There are lots of way to do this, here's one of them

$A(t) = \mathrm{Re}(\tilde{A} e^{-i\omega t}) = \tilde{A}' \cos\omega t + \tilde{A}'' \sin\omega t$
$B(t) = \mathrm{Re}(\tilde{B} e^{-i\omega t}) = \tilde{B}' \cos\omega t + \tilde{B}'' \sin\omega t$
(where $\tilde{A} = \tilde{A}' + i\tilde{A}''$, $\tilde{B} = \tilde{B}' + i\tilde{B}''$)

$$\therefore \langle A(t) B(t)\rangle = \tilde{A}'\tilde{B}'\langle\cos^2\omega t\rangle + \tilde{A}''\tilde{B}''\langle\sin^2\omega t\rangle + (\tilde{A}'\tilde{B}'' + \tilde{A}''\tilde{B}')\langle\cos\omega t \sin\omega t\rangle$$

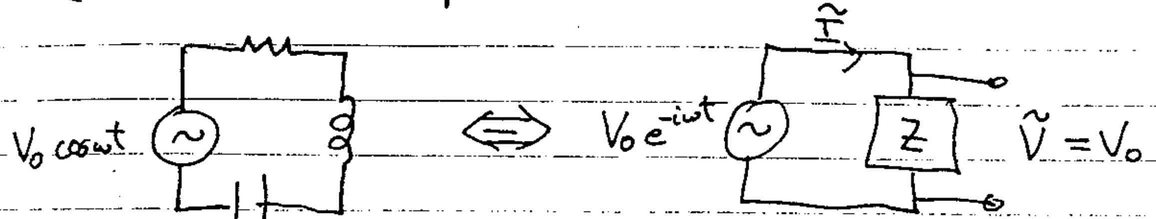
Now $\left.\begin{array}{l}\langle\cos^2\omega t\rangle = \langle\sin^2\omega t\rangle = \frac{1}{2} \\ \langle\cos\omega t \sin\omega t\rangle = \frac{1}{2}\langle\sin 2\omega t\rangle = 0\end{array}\right\}$ check these

$$\Rightarrow \langle A(t) B(t)\rangle = \frac{1}{2}(\tilde{A}'\tilde{B}' + \tilde{A}''\tilde{B}'')$$

$$= \frac{1}{2}\mathrm{Re}(\tilde{A}\tilde{B}^*) = \frac{1}{2}\mathrm{Re}(\tilde{A}^*\tilde{B})$$

QED

(b) Power dissipated in an RLC circuit



$$Z = R - i\left(\omega L - \frac{1}{\omega C}\right)$$

From part (a)

$$\langle P(t)\rangle = \langle V(t)\, I(t)\rangle = \frac{1}{2}\mathrm{Re}\left(\tilde{V}^* \tilde{I}\right)$$

Here $\tilde{V}$ and $\tilde{I}$ are the complex amplitudes of the voltage and current across the load

$$\tilde{V} = V_0 \qquad \tilde{I} = \frac{\tilde{V}}{Z} = \frac{V_0}{Z}$$

$$\therefore \langle P\rangle = \frac{1}{2}\mathrm{Re}\left(\frac{V_0^2}{Z}\right) = \frac{V_0^2}{2}\mathrm{Re}\left(\frac{1}{R - i(\omega L - \frac{1}{\omega C})}\right)$$

$$= \frac{V_0^2}{2}\mathrm{Re}\left(\frac{R + i(\omega L - \frac{1}{\omega C})}{R^2 + (\omega L - \frac{1}{\omega C})^2}\right)$$ Multiplying top and bottom by $R + i(\omega L - \frac{1}{\omega C})$

$$= \frac{V_0^2}{2}\,\frac{R}{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

$$= \frac{V_0^2}{2R}\,\frac{\frac{R^2}{L^2}\omega^2}{\omega^2 \frac{R^2}{L^2} + (\omega^2 - \frac{1}{LC})^2}$$ (multiplying top and bottom by $\frac{\omega^2}{L^2}$)

$$= \frac{V_0^2}{2R}\,\frac{\Gamma^2\omega^2}{\omega^2\Gamma^2 + (\omega^2 - \omega_0^2)}$$

where $\Gamma \equiv \frac{R}{L}$ $\quad \omega_0 = \frac{1}{LC}$

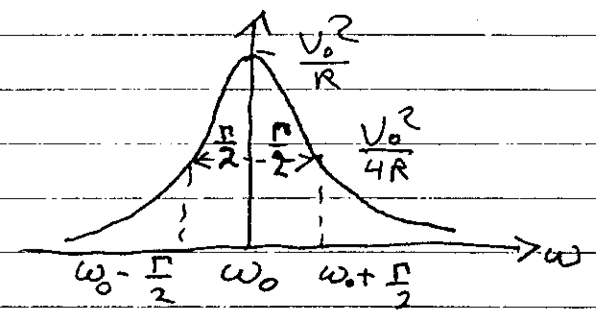
Now in the near resonance approximation

$$\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) \approx 2\omega(\omega - \omega_0)$$

$$\Rightarrow \langle P \rangle = \frac{V_0^2}{2R} \frac{\Gamma^2 \omega^2}{\omega^2 \Gamma^2 + 4\omega^2(\omega - \omega_0)^2}$$

$$\boxed{\langle P \rangle = \frac{V_0^2}{2R} \frac{\Gamma^2/4}{(\omega - \omega_0)^2 + \frac{\Gamma^2}{4}}}$$

Near the resonance, the ⟨Power⟩ dissapated looks lik
a Lorentzian



FWHM $= \Gamma$

(c) The average stored energy is the sum of energies in C +

$$\langle E \rangle = \langle E_c \rangle + \langle E_L \rangle = \frac{1}{2} C \langle V_c^2 \rangle + \frac{1}{2} L \langle I_L^2 \rangle$$

But $\langle E_c \rangle = \langle E_L \rangle$ (capacitor + inductor store equal energies)

$$\Rightarrow \langle E \rangle = 2\langle E_L \rangle = L \langle I_L^2 \rangle$$

$$= L \left(\frac{Re}{2} (\tilde{I} \tilde{I}^*) \right) = \frac{L}{2} Re \left(\frac{\tilde{V}}{Z} \frac{\tilde{V}^*}{Z^*} \right)$$

$$= \frac{L}{2} \frac{V_0^2}{|Z|^2}$$

On resonance $Z = R \Rightarrow \langle E(\omega = \omega_0) \rangle = \frac{L V_0^2}{2R^2}$

On resonance, the average power dissipated is

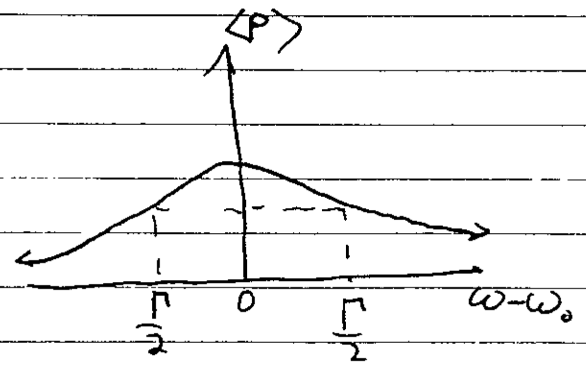
$$\langle P \rangle \big|_{\omega=\omega_0} = \frac{V_0^2}{2R}$$

$$\therefore \; Q \equiv \omega_0 \frac{\langle E \rangle}{\langle P \rangle} = \omega_0 \frac{\frac{L V_0^2}{2R^2}}{\frac{V_0^2}{2R}}$$

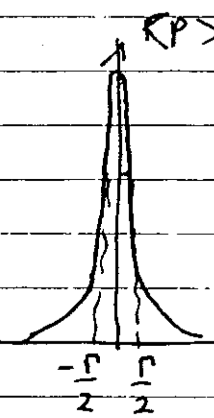
$$\Longrightarrow \boxed{Q = \omega_0 \frac{L}{R} = \frac{\omega_0}{\Gamma}}$$

This is the general form of Q : ratio of oscillation frequency to the damping rate.

It measures the number of periods oscillated before the stored energy decays to $1/e$ of its value. For a good oscillator Q is very large ($\sim 10^4$). The resonance is then very narrow (i.e. if we want to deliver power to the load we must be almost exactly on resona ;



Low Q
(R big)



High Q
(R small)