Homework #2 *solutions*
(Note that all Griffiths Problems are from 4th Ed.)

1) Validity of Ohm's Law:

Area A

Separation S

Anode potential V_0

Cathode potential 0

Charge density $\rho(x) = -en(x)$

Current density $\vec{J} = \rho v \hat{e}_x = -env\hat{e}_x$

Charge conservation $0 = \nabla \cdot \vec{J} = \frac{dJ_x}{dx}$

$\longrightarrow \quad J_x = J = -I/A = \rho v = \text{constant}$ so that $I > 0$

(i) $V(x) = 0$ at $x = 0$; $V(x) = V_0$ at $x = S$

(ii) Poisson equation: $-\rho/\epsilon_0 = \nabla^2 V = \frac{d^2V}{dx^2}$

(iii) Charge conservation: $J = \rho v = -I/A = \text{constant}$

(iv) Electron dynamics: $\frac{1}{2}m_e v^2 = eV(x) \Rightarrow v = \sqrt{\frac{2eV}{m_e}}$

(v) Space-charge limit: $E_x = -\frac{dV}{dx} = 0$ at $x = 0$

(a) $$\frac{d^2V}{dx^2} = -\frac{\rho}{\epsilon_0} = +\frac{I}{A\epsilon_0}\frac{1}{v} = \underbrace{\left(+\frac{I}{A\epsilon_0}\sqrt{\frac{m_e}{2e}}\right)}_{\equiv K > 0 \text{ because } I > 0} V^{-1/2} = +\frac{K}{\sqrt{V}}$$

(b) Get a 1st integral:

$$0 = \underbrace{\frac{dV}{dx}\frac{d^2V}{dx^2}}_{\frac{1}{2}\frac{d}{dx}\left(\frac{dV}{dx}\right)^2} - K\underbrace{V^{-1/2}\frac{dV}{dx}}_{2\frac{d}{dx}V^{1/2}} = \frac{d}{dx}\left(\frac{1}{2}\left(\frac{dV}{dx}\right)^2 - 2KV^{1/2}\right)$$

This implies that

$$\frac{1}{2}\left(\frac{dV}{dx}\right)^2 - 2K V^{1/2} = \text{constant} = 0$$

by the boundary conditions (i) and (v) at $x=0$

$$\therefore \quad \frac{dV}{dx} = 2\sqrt{K}\, V^{1/4}$$

we know we take positive square root because V is positive by Condition (iv) and V must increase with x to match the boundary conditions in Condition (ii).

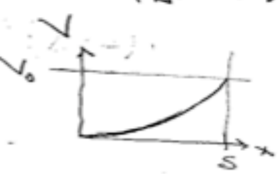
OR

$$\frac{dV}{V^{1/4}} = 2\sqrt{K}\, dx$$

Integrate and apply BC(i) at $x=0$:

$$\boxed{\frac{4}{3} V^{3/4} = 2\sqrt{K}\, x}$$

$$V = \left(\frac{3}{2}\sqrt{K}\, x\right)^{4/3}$$



potential $V \propto x^{4/3}$

velocity $v \propto x^{2/3}$

charge density $\rho \propto x^{-2/3}$

Charge density diverges at $x=0$, but the total charge is finite.

(c) Evaluate at $x=s$:

$$\frac{4}{3} V_0^{3/4} = 2\sqrt{K}\, s$$

$$V_0^{3/4} = \frac{3}{2}\sqrt{K}\, s$$

(Square) $$V_0^{3/2} = \frac{9}{4} s^2 K = \frac{9}{4} s^2 \frac{1}{A\epsilon_0}\sqrt{\frac{m_e}{2e}}\; I$$

$$\boxed{V_0^{3/2} = \frac{9}{4}\,\frac{s^2}{A\epsilon_0}\sqrt{\frac{m_e}{2e}}\; I}$$

$E_x = -\frac{\partial V}{\partial x} \propto x^{1/3}$

E_x is not a constant because space charge changes field from that of a capacitor

Comments: How to think about the differential equation

$$\frac{d^2V}{dx^2} = +\frac{K}{\sqrt{V}}$$

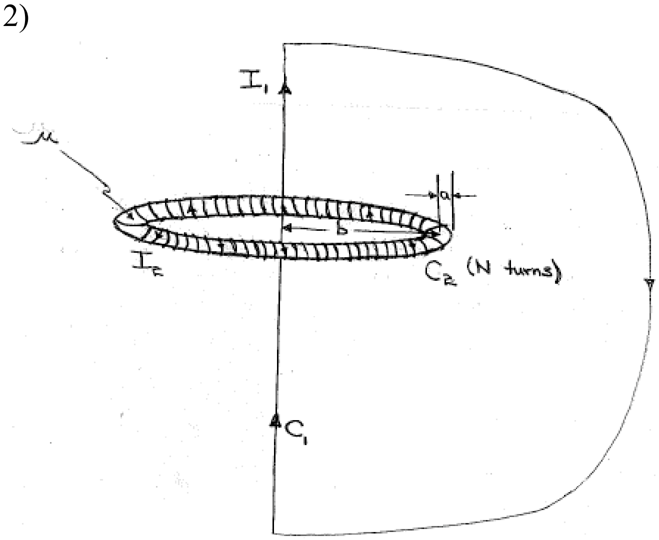
If we think of V as a "position" and x as a "time", then the left side is an "acceleration," and the equation is $F = ma$ with $m = 1$ and a "force" $F = K/\sqrt{V}$. The "force" can be derived from a "potential," i.e.,

$$F = \frac{K}{\sqrt{V}} = -\frac{d}{dV}\underbrace{\left(-2KV^{1/2}\right)}_{\text{"potential"}}$$

The first integral is just the "conservation of energy"

$$\underbrace{\frac{1}{2}\left(\frac{dV}{dx}\right)^2}_{\text{"kinetic energy"}} \underbrace{- 2KV^{1/2}}_{\text{"potential energy"}} = \text{constant}$$

2)



(a) The current I_1 produces an $\vec{H}$-field $\vec{H} = H(r)\,\hat{e}_\phi$

Ampere's Law gives

$$H(r)\,2\pi r = \oint_C \vec{H}\cdot\vec{d\ell} = I_1 \Rightarrow \vec{H} = \frac{I_1}{2\pi r}\hat{e}_\phi.$$

Within iron core, $\vec{B} = \mu\vec{H} = \frac{\mu I_1}{2\pi r}\hat{e}_\phi = \frac{\mu I_1}{2\pi b}\hat{e}_\phi$

neglect variation of $\vec{B}$ across torus ($r = b$)

$$\Phi_2 = N\int_{\text{one loop}} \vec{B}\cdot\underbrace{\vec{da}}_{da\,\hat{e}_\phi} = NB\pi a^2$$

$$\boxed{\Phi_2 = \frac{\mu N a^2}{2b} I_1}$$

(b) The current I_2 produces an $\vec{H}$-field $\vec{H} = H(r)\,\hat{e}_\phi$ only within the torus. Ampere's Law gives

$$H(r)\,2\pi r = \oint_C \vec{H}\cdot\vec{d\ell} = NI_2 \Rightarrow \vec{H} = \frac{NI_2}{2\pi r}\hat{e}_\phi$$

(within torus)

The $\vec{B}$-field is

$$\vec{B} = \mu\vec{H} = \frac{\mu N I_2}{2\pi b}\hat{e}_\phi \quad \text{(within torus)}$$

neglect variation of $\vec{B}$ across torus ($r = b$)

$$\therefore\ \Phi_1 = \int_{S_1} \vec{B}\cdot\underbrace{\vec{da}_1}_{da_1\,\hat{e}_\phi} = B\pi a^2$$

$$\boxed{\Phi_1 = \frac{\mu N a^2}{2b} I_2}$$

(c) $$M = \frac{\Phi_1}{I_2} = \frac{\Phi_2}{I_1} = \boxed{\frac{\mu N a^2}{2b} = M}$$

(d) Emf induced in C_1 is

$$\mathcal{E}_1 = -\frac{d\Phi_1}{dt} = -M\frac{dI_2}{dt}$$

The resulting current is

$$I_1 = \frac{\mathcal{E}_1}{R} = -\frac{M}{R}\frac{dI_2}{dt}$$

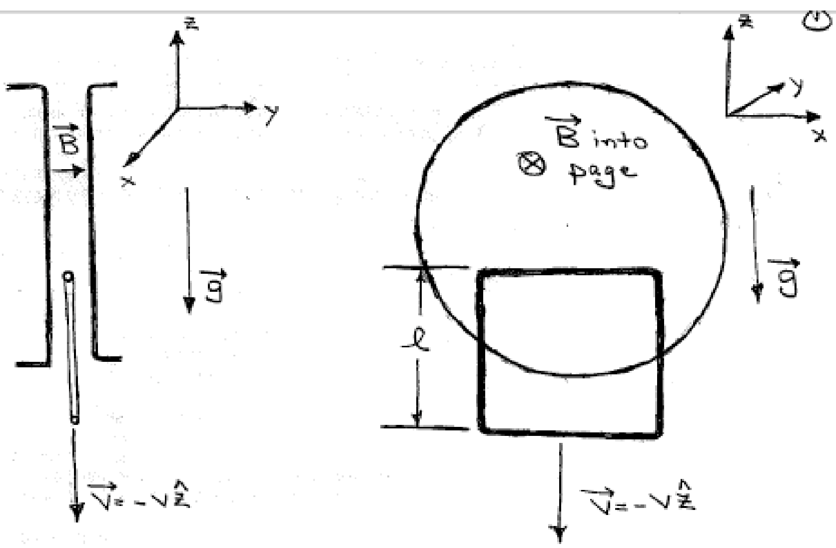
Lenz's law: current flows in negative direction to oppose increase in flux through C_1

$$Q_1 = \int_0^\infty dt\,\frac{dQ}{dt} = -\frac{M}{R}\underbrace{\int_0^\infty dt\,\frac{dI_2}{dt}}_{I_2(\infty) - I_2(0) = I_f} = -\frac{M I_f}{R}$$

$I_2(0) \to 0$

$$\boxed{Q_1 = -\frac{M I_f}{R} = -\frac{\mu N a^2}{2bR} I_f}$$

3)



(a) $$M = \rho_m \underbrace{(4\ell A)}_{\text{volume of frame}} = \boxed{4\ell A\rho_m = M}$$

$$\boxed{R = \frac{4\ell}{\sigma A}}$$

(b)
$$\mathcal{E} = \oint_C \vec{v}\times\vec{B}\cdot d\vec{\ell}$$

Only the top contributes
$$= \int_{top} \underbrace{\vec{v}\times\vec{B}}_{-v\hat{z}\times B\hat{y} = vB\hat{x}}\cdot \underbrace{d\vec{\ell}}_{dx\,\hat{x}} = vB\int_0^\ell dx = \boxed{vB\ell = \mathcal{E}}$$

$$I = \frac{\mathcal{E}}{R} = \boxed{\frac{vB\ell}{R} = I}$$

The current flows clockwise since positive charges are pushed in the +x direction in the top of the frame

(c)
$$\vec{F}_{mag} = I\oint d\vec{\ell}\times\vec{B} = I\int_{top} \underbrace{d\vec{\ell}\times\vec{B}}_{dx\hat{x}\times B\hat{y} = B\,dx\,\hat{z}} = IB\hat{z}\underbrace{\int_0^\ell dx}_{\ell}$$

right and left sides cancel; bottom doesn't contribute because $\vec{B}=0$; so only the top contributes

$$\boxed{\vec{F}_{mag} = IB\ell\hat{z} = v\frac{B^2\ell^2}{R}\hat{z}}$$

(d) Newton's 2nd Law: $\vec{F} = M\vec{a}$

$\vec{v} = -v(t)\hat{z}$

$$-M\frac{dv}{dt}\hat{z} = M\frac{d\vec{v}}{dt} = M\vec{a} = \vec{F} = \vec{F}_g + \vec{F}_{mag} = -Mg\hat{z} + v\frac{B^2\ell^2}{R}\hat{z}$$

$$\boxed{\frac{dv}{dt} = g - v\,\frac{B^2\ell^2}{MR}}$$

Terminal velocity: $0 = \frac{dv}{dt} \Rightarrow g = v_c \frac{B^2\ell^2}{MR}$

$$\boxed{v_c = \frac{MgR}{B^2\ell^2}}$$

e-folding time: $\boxed{\tau = \frac{MR}{B^2\ell^2} = \frac{v_c}{g}}$

<u>Solution:</u> $\frac{d}{dt}(v - v_c) = \frac{1}{\tau}(v_c - v) = -\frac{1}{\tau}(v - v_c)$

$$v(t) - v_c = (\underbrace{v_0}_{=0} - v_c)\, e^{-t/\tau}$$

$= 0$ (dropped from rest)

$$\boxed{v(t) = v_c\left(1 - e^{-t/\tau}\right)}$$

(e). Energy conservation means that

$$\frac{d}{dt}(\underbrace{T}_{\text{kinetic energy}} + \underbrace{U}_{\text{gravitational potential energy}}) = -\left(\begin{matrix}\text{power dissipated} \\ \text{in } R\end{matrix}\right) = -I^2R = \frac{\mathcal{E}^2}{R}$$

Once the frame is at terminal velocity, T is constant, so $dT/dt = 0$.

$$\frac{dU}{dt} = \frac{d}{dt}(Mgz) = Mg\underbrace{\frac{dz}{dt}}_{-v_c} = -Mgv_c$$

$$\therefore\quad Mgv_c = I^2R = \frac{\mathcal{E}^2}{R} = \frac{v_c^2B^2\ell^2}{R}$$

$$\Rightarrow \boxed{v_c = \frac{MgR}{B^2\ell^2}} \checkmark$$

This says that the rate at which gravity does work on the frame equals the rate at which energy is dissipated in R

(f) $$\frac{MR}{\ell^2} = \frac{4\ell A\rho_m 4\ell}{\sigma A\ell^2} = \frac{16\rho_m}{\sigma}$$

$$\boxed{\begin{aligned} \tau &= \frac{MR}{B^2\ell^2} = \frac{16\rho_m}{B^2\sigma} \\ v_c &= g\tau = \frac{16\rho_m g}{B^2\sigma} \end{aligned}}$$

A good conductor has $\sigma = \alpha \times 10^7 \frac{1}{\Omega\text{-m}}$

order unity

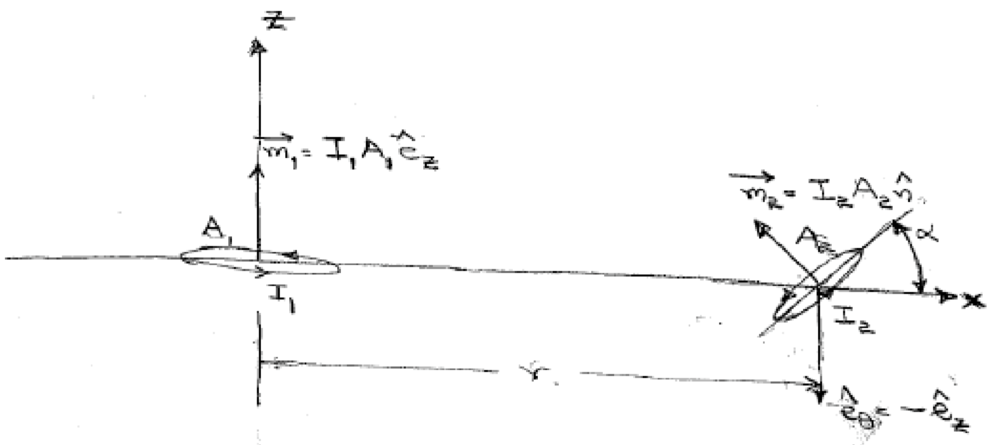
All solids have $\rho_m = \beta\ g/cm^3 = \beta \times 10^3\ kg/m^3$

order unity

$$\tau = \frac{16\,\beta \times 10^3\ kg/m^3}{(1\,T)^2\,(\alpha \times 10^7/\Omega\text{-m})} = \frac{\beta}{\alpha} \times 16 \times 10^{-4}$$

$$\boxed{\begin{aligned} &\tau = 1.6\,\frac{\beta}{\alpha}\times 10^{-3}\ \text{s} \sim 1\ \text{ms} \\ &v_c = \underset{\underset{9.8\ \text{m/s}^2}{\uparrow}}{g}\,\tau \sim 10^{-2}\ \text{m/s} \end{aligned}}$$

4)



r big enough that fields can be dealt with in the dipole approximation.

Dipole field of loop 1: $\vec{A}_1 = \frac{\mu_0}{4\pi}\frac{\vec{m}_1 \times \hat{e}_r}{r^2}$, $\vec{B}_1 = \frac{\mu_0 m_1}{4\pi}\frac{2\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta}{r^3}$

(a) Magnetic field due to loop 1 at loop 2.

$$\vec{B}_1 = \frac{\mu_0 m_1}{4\pi r^3}\hat{e}_\theta = -I_1 \frac{\mu_0 A_1}{4\pi r^3}\hat{e}_z$$

ignore variation of $\vec{B}_1$ across surface of loop 2

$$\Phi_2 = \vec{B}_1 \cdot A_2 \hat{n} = -I_1 \frac{\mu_0 A_1 A_2}{4\pi r^3}\underbrace{\hat{e}_z \cdot \hat{n}}_{\cos\alpha}$$

$$\boxed{\Phi_2 = -I_1 \frac{\mu_0 A_1 A_2 \cos\alpha}{4\pi r^3}}$$

(2)

(b) $$\boxed{M = M_{21} = M_{12} = \frac{\Phi_2}{I_1} = -\frac{\mu_0 A_1 A_2 \cos\alpha}{4\pi r^3}}$$

(c) $\vec{F} = I_1 I_2 \nabla M$

$$F_x = I_1 I_2 \frac{\partial M}{\partial x}$$

$$= \frac{\partial M}{\partial r} = \frac{3\mu_0 A_1 A_2 \cos\alpha}{4\pi r^4}$$

$$\boxed{F_x = \frac{3\mu_0 m_1 m_2 \cos\alpha}{4\pi r^4}}$$

$\alpha = 0$: repulsive
$\alpha = \pi/2$: $F_x = 0$
$\alpha = \pi$: attractive

General force on a dipole:
$\vec{F} = I_1 I_2 \nabla M = I_2 \nabla \Phi_2$
$\vec{F} = I_2 \nabla(\vec{B}_1 \cdot A_2 \hat{n}_2) = \nabla(\vec{B}_1 \cdot \vec{m}_2)$

OR

$\vec{F} = \nabla(\vec{m}_2 \cdot \vec{B}_1)$

$$F_x = \frac{\partial}{\partial x}(\vec{m}_2 \cdot \vec{B}_1) = -\frac{\partial}{\partial x}\left(m_2 \hat{n}_2 \cdot \frac{\mu_0 m_1}{4\pi r^3}\hat{z}\right)$$

$$= -\frac{\partial}{\partial x}\left(+\frac{\mu_0 m_1 m_2 \cos\alpha}{4\pi r^3}\right)$$

$$\boxed{F_x = \frac{3\mu_0 m_1 m_2 \cos\alpha}{4\pi r^4}}$$

(d) $$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -I_1 \frac{dM}{dt}$$ $\alpha = \omega t$

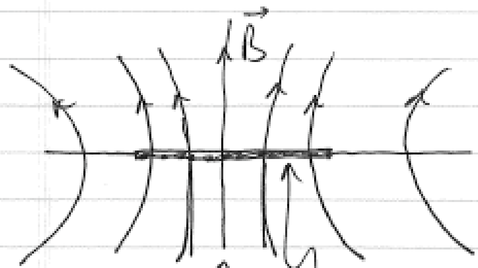
$$\frac{dM}{dt} = \frac{\omega \mu_0 A_1 A_2 \sin\omega t}{4\pi r^3}$$

$$\boxed{\mathcal{E}_2 = -I_1 \frac{\omega \mu_0 A_1 A_2 \sin\omega t}{4\pi r^3}}$$

$\omega t = \pi/2$: emf opposite drawing
$\omega t = \pi$: emf 0
$\omega t = 3\pi/2$: emf in direction of drawing

5)
Betatron

Side View

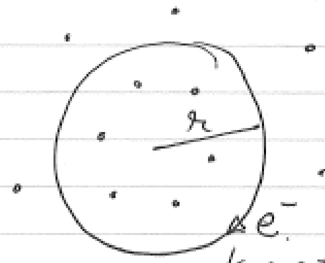


Note the non-uniform $\vec{B}$ which, however, is symmetric about central axis

axis

Plane of e^- trajectory

Top View



e^- trajectory

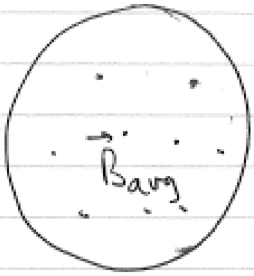
Dots indicate $\vec{B}$ field (out of page)

We want the e^- to remain at constant radius r while increasing $\vec{B}$ which will be used to accel. the e^- via an induced $\vec{E}$ at r.

$$\underbrace{\oint_C \vec{E}\cdot d\vec{l}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{a}$$

$E\,2\pi r$ because, by symmetry, $\vec{E}$ is constant and tangential at r.

R.H.S. is hard because $\vec{B}$ is non-uniform over surface. Let's use the average $\vec{B}$, B_{avg}, over the area of the plane of the trajectory:



$$\oint_S \vec{B}\cdot d\vec{a} = \underbrace{\oint_S B\,da}_{\text{always true}} \cong \oint_S B_{avg}\,da$$

$$= B_{avg}\,\pi r^2$$

$$\Rightarrow \quad \frac{d}{dt}\oint_S \vec{B}\cdot d\vec{a} \cong \pi r^2 \frac{dB_{avg}}{dt}$$

and
$$\boxed{E = \frac{r}{2}\frac{dB_{avg}}{dt}}$$

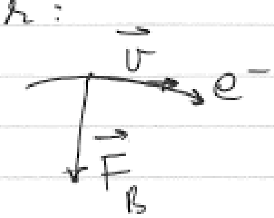
Now we require that e^- stays in circular orbit of radius r, while feeling a tangential force that accelerates it around the circle:

$$\boxed{\frac{dp}{dt} = qE = \frac{qr}{2}\frac{dB_{avg}}{dt}}$$ (here, p is magnitude of its momentum)

Integrating, we see how the momentum changes as a function of B_{avg}:

$$p = p_0 + \frac{qr}{2}\Delta B_{avg}$$ (we'll take p_0, initial e^- momentum, $= 0$)

Circular orbit: here we use the $\vec{B}$ force at r:



$$F_B = qvB(r) \quad (\vec{v} \perp \vec{B})$$
$$= \frac{dp_\perp}{dt}$$

where $p_\perp$ is the transverse momentum.

Recall that for circular motion:

$$a = \frac{v^2}{r} \rightarrow \text{or, using } \omega = \frac{v}{r} \text{ (angular velocity)}$$

we can write:

$$\frac{dp_\perp}{dt} = \frac{mv^2}{r} = \omega p$$ ← total mom.

So:

$$\frac{dp_\perp}{dt} = \boxed{q\, v \underbrace{\Delta B(r)}_{\Delta B \text{ at orbit, which also increases with time.}} = \omega p = \frac{v}{r} p}$$

Solving for p: $\boxed{p = q\, r \Delta B(r)}$

and comparing with earlier eqn: $p = \frac{qr}{2} \Delta B_{avg}$

$$\rightarrow \quad q r \Delta B(r) = \frac{qr}{2} \Delta B_{avg}$$

$$\Rightarrow \boxed{\Delta B_{avg} = 2 \Delta B(r)}$$

Average B over orbit (inside r) is equal to twice B at orbit!

(b) By looking at rate at which $\vec{E}$ does <u>work</u> on e^-, calculate the d(K.E.)/dt of the e- at radius R in terms of R, B, dB/dt, q, and m.

$$\frac{d(K.E.)}{dt} = q E v \quad (\vec{E} \parallel \vec{v})$$

$$= \frac{qR}{2} \frac{dB_{avg}}{dt} \cdot v$$

Using $\frac{dp_\perp}{dt} = \frac{mv^2}{R} = q v B(R)$

$$\rightarrow \quad v = \frac{Rq}{m} B(R)$$

We get:

$$\frac{d(K.E.)}{dt} = \frac{qR}{2}\frac{dB_{avg}}{dt} \cdot \frac{RqB(R)}{m}$$

$$= \frac{q^2R^2}{2m}\frac{dB_{avg}}{dt} \cdot B(R)$$

Using $\frac{dB_{avg}}{dt} = 2\frac{dB(R)}{dt}$:

$$\boxed{\frac{d(K.E.)}{dt} = \frac{q^2R^2}{m}\frac{dB(R)}{dt} \cdot B(R) = \frac{q^2R^2}{2m}\frac{dB^2(R)}{dt}}$$