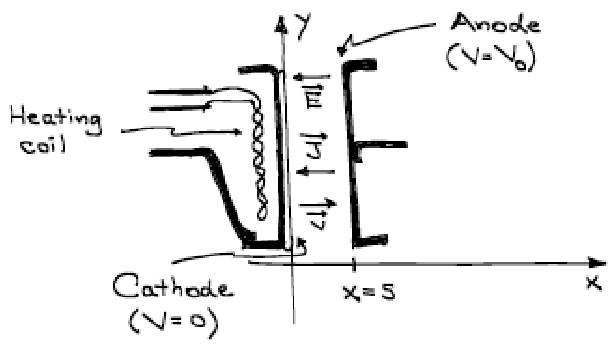
Homework #2 *Due Friday 9/5/2014 by NOON in my mailbox*
(Note that all Griffiths Problems are from 4th Ed.)

1) Checking the validity of Ohm's Law:

Consider the plane diode (two-electrode vacuum tube) shown in the drawing below. One electrode, the cathode, is coated with a material that emits electrons copiously when heated. The other electrode, the anode, is simply a metal plate. Both electrodes have area A, and they are separated by a distance s. By means of a battery the anode is maintained at positive potential V_0 relative to the (grounded) cathode. Electrons emerge from the heated cathode with very low velocities and are accelerated toward the positive anode. In the space between the cathode and the anode the current consists of these moving electrons. The circuit is completed by current flow in external wires, with which we are not concerned here.



We are interested in the steady-state (time independent) "space-charge-limited" behavior of the diode. We assume that the transverse dimensions of the electrodes are much larger than s; thus we neglect edge effects and assume that all quantities depend only on the x coordinate. The boundary conditions on the potential give us

$$\text{Condition (i):} \quad V(x) = 0 \text{ at } x = 0 \text{ and } V(x) = V_0 \text{ at } x = s.$$

If $n(x)$ is the number density of electrons at x, then the charge density is $\rho(x) = -en(x)$. The current density $\mathbf{J} = J_x \hat{\mathbf{e}}_x$ has only an x component $J_x = \rho(x)v(x)$, where $v(x)$ is the velocity of the electrons at x. Charge conservation in this (time independent) situation simplifies to $0 = \nabla \cdot \mathbf{J} = dJ_x/dx$, which gives us

$$\text{Condition (ii):} \quad J_x = \rho v = -I/A \quad \text{is constant.}$$

Here I is the current flowing between the two electrodes. (The sign change is made so that I is a positive quantity.)

We want to solve for the relation between current I and voltage V_0 when the current flow is "space-charge-limited," which means that there is no limitation on the number of electrons the cathode can supply. The cathode supplies more and more electrons until the field created by the "space charge" of these electrons becomes so large that it prevents further increases in the "space-charge density" of electrons. In this case we must take into account the field created by the electrons, in addition to the field due to the potential difference between the electrodes. Thus we must use the Poisson equation, which gives us

$$\text{Condition (iii):} \quad -\frac{\rho}{\epsilon_0} = \nabla^2 V = \frac{d^2V(x)}{dx^2}.$$

Assume that the electrons leave the cathode with zero velocity (their velocity is actually very small compared to the velocity they gain in the electric field, but it is, of course, not exactly zero). Then the kinetic energy $\frac{1}{2}m_e v^2(x)$ of each electron at x (m_e is the mass of the electron) is given by the electric potential energy $eV(x)$. Thus electron dynamics (conservation of electron energy) gives us

$$\text{Condition (iv):} \quad \tfrac{1}{2}m_e v^2(x) = eV(x).$$

We need one more condition to specify the problem (you might want to consider why). Space-charge-limited current flow implies that the electrons modify the field so much that the electric field $\mathbf{E} = E_x\hat{\mathbf{e}}_x$ at the cathode is zero. If E_x at $x = 0$ were negative, then more electrons would be pulled off the cathode until E_x at $x = 0$ became zero, and if E_x at $x = 0$ were positive, then electrons would be pushed onto the cathode until E_x at $x = 0$ became zero. Thus space-charge-limited current flow gives us

$$\text{Condition (v):} \quad 0 = E_x = -\frac{dV}{dx} \text{ at } x = 0.$$

Conditions (i)–(v) can be used to solve for the current voltage relationship.

(a) Use conditions (ii), (iii), and (iv) to *show* that the potential $V(x)$ obeys the differential equation

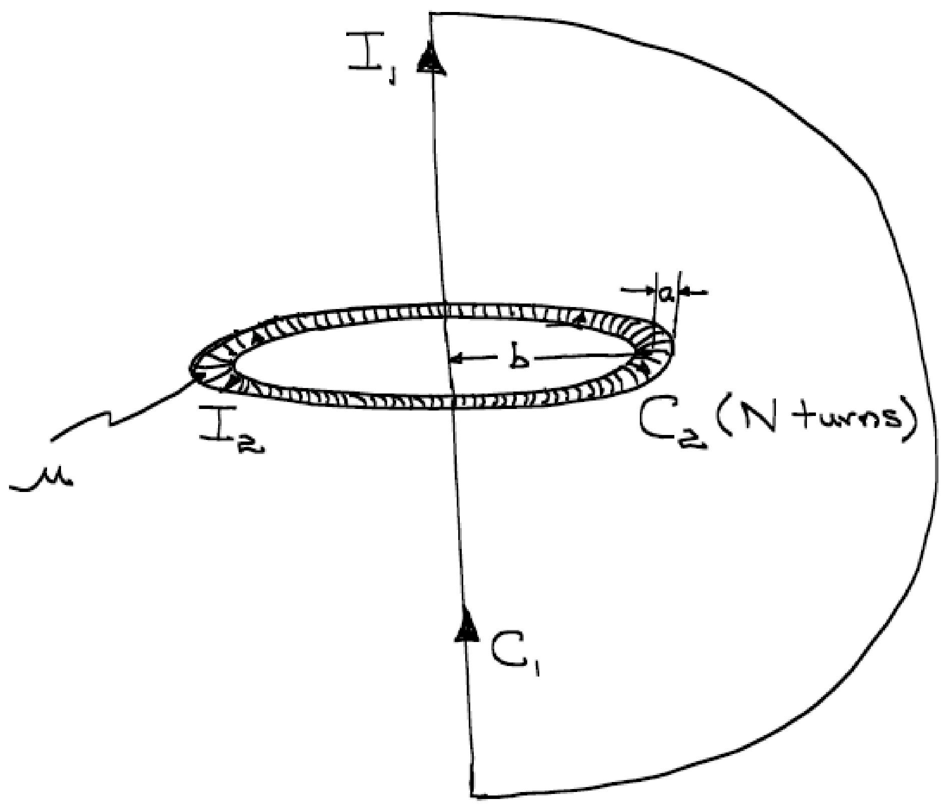
$$\frac{d^2V(x)}{dx^2} = \frac{K}{\sqrt{V}}, \tag{1}$$

where K is a positive constant. *Find* the constant K in terms of the various parameters of the problem.

(b) *Solve* for $V(x)$ by integrating Eq. (1) using the boundary conditions (i) and (v). [Hint: Find a first integral of Eq. (1) by multiplying both sides of the equation by $2\,dV/dx$; the left side then becomes $(d/dx)\big((dV/dx)^2\big)$, and the right side becomes $(d/dx)(4K\sqrt{V})$.]

(c) Evaluate the solution of part (b) at $x = s$ to *show* that the current I is proportional to $V_0^{3/2}$. *Write* down the proportionality constant. This dependence of charge on voltage is called the *Child-Langmuir law*.

2) Consider a toroidal piece of iron which has magnetic permeability μ. As shown in the drawing, the torus has outer (large) radius b and inner (small) radius a; its cross-sectional area is thus πa^2. The torus is closely wound with N turns of wire which make up a closed loop C_2 (as shown, C_2 goes down on the outside of the torus and up on the inside). A long straight wire lies along the axis of symmetry of the torus. The two ends of the long straight wire are joined far away from the torus by a half-circle of wire, thus making a closed loop C_1.



(a) Suppose a current I_1 flows in loop C_1 as shown. *Find* the resulting magnetic field $\mathbf{B}$ within the iron torus. You are to assume, first, that the straight part of C_1 is so long that you can treat it as infinitely long and neglect the contribution of the half-circle and, second, that a is so small compared to b that you can neglect the variation of $\mathbf{B}$ across the torus. *Find* the flux Φ_2 through C_2 due to the current I_1.

(b) Suppose now that a current I_2 flows in loop C_2 as shown. *Find* the resulting magnetic field $\mathbf{B}$ within the iron torus. Again you are to assume that a is so small compared to b that you can neglect the variation of $\mathbf{B}$ across the torus. *Find* the flux Φ_1 through C_1 due to the current I_2.

(c) What is the mutual inductance M between C_1 and C_2?

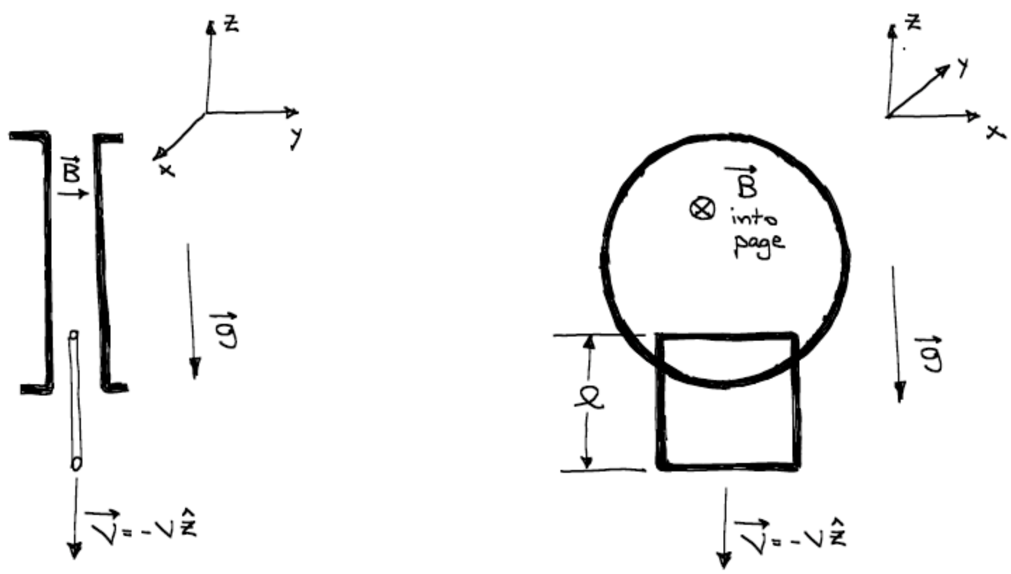
(d) Suppose now that loop C_1 has resistance R and that the current in C_2 is changed from $I_2 = 0$ to a final value of $I_2 = I_f$. Under these circumstances, how much charge Q_1 flows along C_1 in the direction shown? Neglect the self-inductance of C_1. Be sure you get the sign right.

3) HALF for Xtra-credit

Consider a square metal frame whose sides have length ℓ and which is made from a wire or rod whose cross-sectional area is A. The material of the wire or rod has mass density ρ_m and electrical conductivity σ.

(a) *Write* expressions for the mass M and the resistance R of the frame. Till part (f), you should express your answers in terms of M and R.

The frame is located, as shown in the drawing below, between the poles of a magnet. The upper side of the frame is in the gap between the poles, where there is a nearly uniform horizontal magnetic field $\mathbf{B} = B\hat{\mathbf{y}}$. The lower side of the frame is outside the magnet gap where the field, though not zero, is negligible for the purposes of this problem. Suppose the frame is released from rest and falls under its own weight. Let $\mathbf{v} = -v(t)\hat{\mathbf{z}}$ $[v(t) \geq 0]$ be the velocity of the frame at time t after its release.



(b) *Find* the motional emf $\mathcal{E}(t)$ induced in the frame and the current $I(t)$ that flows in the frame at time t. Write your answer in terms of $v(t)$. Does the current flow clockwise or counterclockwise in the lower drawing at right.

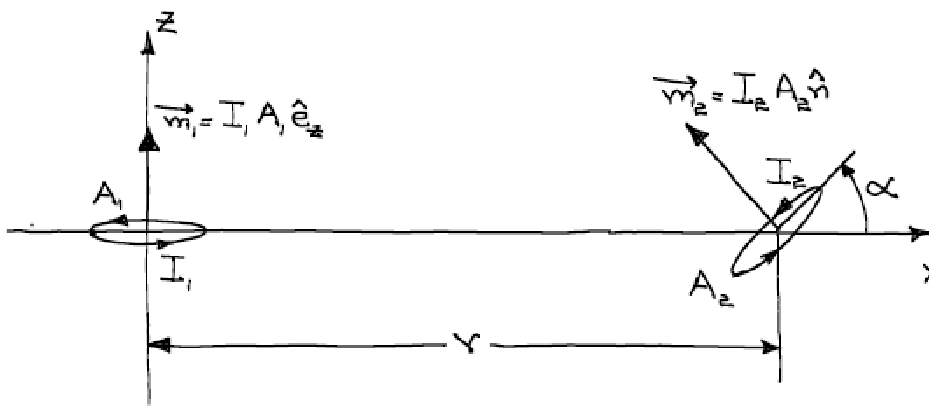
(c) *Find* the magnetic force $\mathbf{F}_{\text{mag}}$ on the frame due to the current $I(t)$. Again write your answer in terms of $v(t)$.

(d) *Write* and *solve* the differential equation for the frame's downward velocity $v(t)$. The frame acquires a terminal velocity v_c after an e-folding time τ. *Find* v_c and τ (you don't have to be able to solve the differential equation to find v_c and τ).

(e) *Interpret* your expression for the terminal velocity in terms of energy conservation.

(f) *Show* that v_c and τ are independent of the size ℓ and cross-sectional area A of the frame. Assuming a strong magnetic field of 1 Tesla, *estimate* v_c and τ for a good conductor.

4) Consider the two current loops shown in the drawing. Loop 1 (on the left) sits at the origin and lies in the plane $z = 0$. It has area A_1, carries current I_1 in the direction shown, and thus has magnetic dipole moment $\mathbf{m}_1 = I_1 A_1 \hat{\mathbf{e}}_z$. Loop 2 (on the right) is a distance r from the origin along the x axis; the plane of loop 2 makes an angle α with the plane $z = 0$. Loop 2 has area A_2, carries current I_2 in the direction shown, and thus has magnetic dipole moment $\mathbf{m}_2 = I_2 A_2 \mathbf{n}$, where $\mathbf{n}$ is a unit vector normal to the plane of loop 2.



Throughout this problem you should assume that the loops are so small and so far apart that (i) you can treat their magnetic fields in the dipole approximation and (ii) you can ignore the variation in the magnetic field produced by one loop across the surface of the other loop.

(a) *Find* the flux Φ_2 through loop 2 due to the magnetic field of loop 1.

(b) *Find* the mutual inductance M between the two loops.

(c) The magnetic interaction energy between the two loops is $W = MI_1I_2$, and the force between the loops is given by $\mathbf{F} = \nabla W = I_1 I_2 \nabla M$. *Find* the x component of the force on loop 2 due to loop 1, and *show* that it has the standard form for the force on a dipole. Is the force attractive or repulsive when $\alpha = 0$? When $\alpha = \pi/2$? When $\alpha = \pi$?

(d) Suppose that loop 2 rotates with angular velocity ω—that is, $\alpha = \omega t$. *Find* the electromotive force $\mathcal{E}_2$ induced in loop 2 due to the field of loop 1. Does the emf drive current flow in the direction shown in the drawing or in the opposite direction when $\alpha = \pi/2$? When $\alpha = \pi$? When $\alpha = 3\pi/2$?

5) Betatron Problem in Griffiths: 7.50