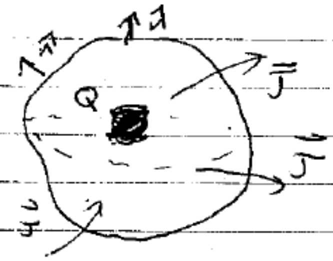
Homework #1 *solutions*
(Note that all Griffiths Problems are from 4th Ed.)

1) Some mathematical review of material that will be useful this semester.

a) Consider a closed volume V



The amount of "Q" inside V changes because Q flows in or out of volume ~~or~~ is created/destroyed

$$\frac{dQ}{dt} = \underset{\text{decreases } Q}{-}(\text{Rate of flow out of } V) + (\text{Rate of creation inside } V)$$

$$= -\int_S \vec{J}\cdot\hat{n}\,da + \int_V R\,d^3r$$

where S is the surface bounding V

Now, according to the divergence theorem

$$\int_S \vec{J}\cdot\hat{n}\,da = \int_V \vec{\nabla}\cdot\vec{J}\,d^3r$$

and if ρ is the local density of Q

$$Q = \int \rho\,d^3r$$

$$\Rightarrow \frac{d}{dt}\int \rho\,d^3r = -\int_V (\vec{\nabla}\cdot\vec{J})\,d^3r + \int_V R\,d^3r$$

$$\Rightarrow \boxed{\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J} + R}$$

This is a local conservation law at any point $\vec{r}$

1b)

(i) $(-1-i)^2 = 1+2i+(-i)^2 = \boxed{2i = 2e^{i\pi/2}}$

another way $(-1-i)^2 = (1+i)^2 = (\sqrt{2}\, e^{i\pi/4})^2$

polar form

$\Rightarrow (-1-i)^2 = 2e^{i\pi/2} = 2i$

(ii) $\dfrac{1}{\omega^2-\omega_0^2-i\omega\Gamma} * \dfrac{(\omega^2-\omega_0^2+i\omega\Gamma)}{(\omega^2-\omega_0^2+i\omega\Gamma)}$ Multiply top and bottom by conjugate

$$\boxed{= \left[\frac{\omega_0^2-\omega_0^2}{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}\right] + i\left[\frac{\omega\Gamma}{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}\right]}$$

Real-imaginary form

$$\frac{1}{\omega^2-\omega_0^2-i\omega\Gamma} = \left(\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}\; e^{i\theta}\right)^{-1}$$

where $\theta = \tan^{-1}\left(\dfrac{-\omega\Gamma}{\omega^2-\omega_0^2}\right)$

$$\boxed{\frac{1}{\omega^2-\omega_0^2-i\omega\Gamma} = \frac{1}{\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}}\; e^{-i\theta}}$$

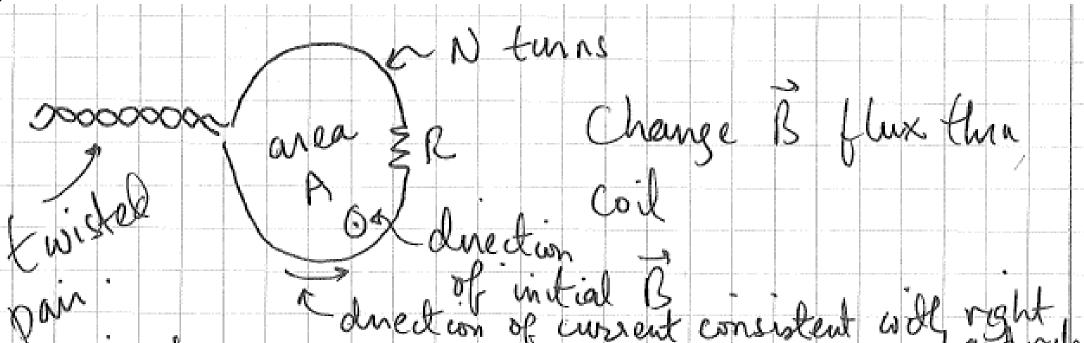
(iii) $\sqrt{-i} = (e^{-i\pi/2})^{1/2}$ $\boxed{= e^{-i\pi/4} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}}$

(iv)
$$\boxed{\frac{e^{-i\omega t}}{\omega^2-\omega_0^2-i\omega\Gamma} = \frac{1}{\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}}\, e^{-i(\omega t+\theta)}}$$

where $\theta = \tan^{-1}\left(\frac{-\omega\Gamma}{\omega^2-\omega_0^2}\right)$

$$= \frac{\cos(\omega t+\theta)}{\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}} - i\,\frac{\sin(\omega t+\theta)}{\sqrt{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}}$$

2)
a)



Change $\vec{B}$ flux thru coil

Initial Φ_B in coil is Φ_i, final value Φ_f. What's the total charge Q which passes thru circuit due ~~to~~ to this $\Delta\Phi_B$ and show that its independent of the $d\Phi_B/dt$.

$$RI = R\frac{dQ}{dt} = \mathcal{E} = -\frac{d\Phi_B}{dt}$$

$$\frac{dQ}{dt} = -\frac{1}{R}\frac{d\Phi_B}{dt} \rightarrow \Delta Q = -\frac{1}{R}\Delta\Phi_B = -\frac{1}{R}(\Phi_f - \Phi_i)$$

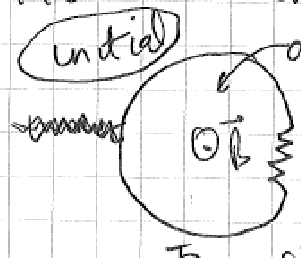
$$\boxed{\Delta Q = \frac{-(\Phi_{B,f} - \Phi_{B,i})}{R}}$$

Sign? Take $\Delta\Phi > 0$ and take + flux to be out of page.

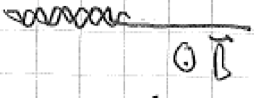
That means that the flux increases in direction out of page, so by Lenz Law the current should flow clock-wise ↻ to give an opposing flux into page ("Nature abhors a change in flux").
The minus sign is there to tell you that the direction of the current is − that which you would get by using the right hand rule with thumb in direction of initial flux.

Now take initial direction of coil & $\vec{B}$ as

b) initial



flip coil by 90°:



$$\Phi_i = \int \vec{B}\cdot d\vec{a} = NBA \qquad \Phi_f = 0$$

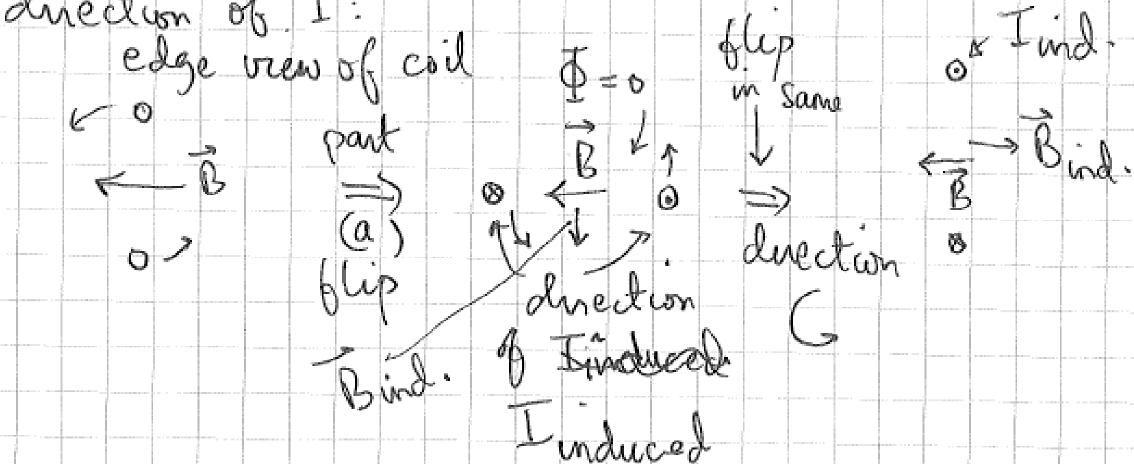
$$\rightarrow \boxed{\Delta Q = \frac{NBA}{R}}$$

with direction of I shown

c) flip coil by 180°
Sum the ΔQ in 2 steps:

part a) result of 90° flip + 2nd 90° flip.

For the latter we need to keep track of ~~dire~~ direction of I:

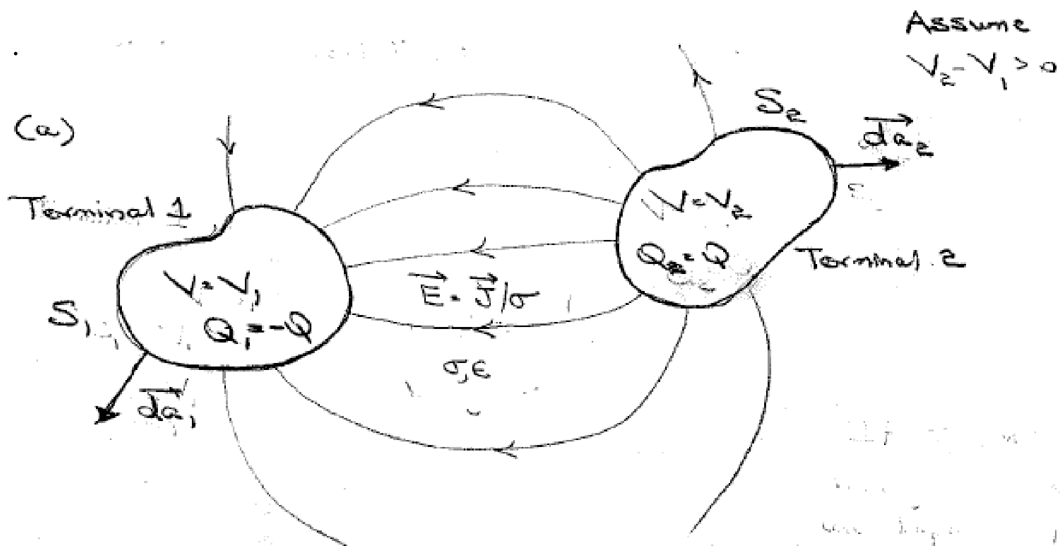


$I_{induced}$ continues to be in same direction

so: $\boxed{\Delta Q = 2NBA/R}$

3)

(a)



culate resistance R:

$$V_2 - V_1 = -\int_1^2 \vec{E}\cdot\vec{dl}$$

$$I = \oint_{S_2} \vec{J}\cdot\vec{da}_2 = -\oint_{S_1} \vec{J}\cdot\vec{da}_1$$

$$= \underbrace{\sigma \oint_{S_2} \vec{E}\cdot\vec{da}_2}_{\text{Ohm's law}} = \frac{\sigma}{\epsilon}\oint_{S_2}\vec{D}\cdot\vec{da}_2 \underset{\text{Gauss's law}}{=} \frac{\sigma}{\epsilon} Q_2 = \frac{\sigma}{\epsilon} Q$$

Notice that for any surface S surrounding terminal 2,

$$I = \oint_S \vec{J}\cdot\vec{da} \underset{\text{Ohm's law}}{=} \frac{\sigma}{\epsilon}\oint_S \vec{D}\cdot\vec{da} \underset{\text{Gauss's law}}{=} \frac{\sigma Q_2}{\epsilon}$$

$$R = \frac{V_2 - V_1}{I} = \frac{\epsilon}{\sigma}\underbrace{\frac{V_2 - V_1}{Q}}_{1/C}$$

(C is the capacitance between the terminals)

$$\Longrightarrow \boxed{RC = \epsilon/\sigma}$$

Units: $[RC] = [\epsilon/\sigma] = s$

$[R] = \Omega = \frac{s}{F}$

$[\sigma] = \frac{1}{\Omega\text{-}m} = \frac{F/m}{s}$

$[\epsilon] = F/m$

(b) The analysis in (a) is consistent with charge conservation because

$$\nabla\cdot\vec{J} = \nabla\cdot(\sigma\vec{E}) = \nabla\cdot\left(\frac{\sigma}{\epsilon}\vec{D}\right) = \frac{\sigma}{\epsilon}\nabla\cdot\vec{D} = 0 \quad \text{within medium}$$

no free charge in medium

σ, ε homogeneous

Things would be different if σ or ε depended on position.

(c)

$$P = \sigma\int d\tau\, E^2 = -\sigma\int d\tau\, \underbrace{\nabla V\cdot\vec{E}}_{\nabla\cdot(V\vec{E}) \,-\, V\underbrace{\nabla\cdot\vec{E}}_{\rho/\epsilon_0 = 0}}$$

integral over medium

no charge in the medium

$$= -\sigma\int d\tau\, \nabla\cdot(V\vec{E})$$

$$= +\sigma\oint_{S_1}\vec{da}_1\cdot V\vec{E} + \sigma\oint_{S_2}\vec{da}_2\cdot V\vec{E}$$

This is the divergence theorem: the outward normals from the medium are $-\vec{da}_1$ and $-\vec{da}_2$; the surface at ∞ doesn't contribute because $V\vec{E}$ goes to zero faster than $1/r^2$

$$= \sigma\left(V_1\oint_{S_1}\vec{E}\cdot\vec{da}_1 + V_2\oint_{S_2}\vec{E}\cdot\vec{da}_2\right)$$

$$= \frac{\sigma}{\epsilon}\left(V_1\underbrace{\oint_{S_1}\vec{D}\cdot\vec{da}_1}_{Q_1 = -Q} + V_2\underbrace{\oint_{S_2}\vec{D}\cdot\vec{da}_2}_{Q_2 = Q}\right)$$

$$= \frac{\sigma}{\epsilon}(V_2 - V_1)\,Q$$

$$= \frac{\sigma}{\epsilon}\,C\,(V_2 - V_1)^2$$

So $P = \underbrace{\frac{\sigma}{\epsilon} C}_{1/R \text{ from part (a)}} (V_2 - V_1)^2$

$$= \frac{(V_2 - V_1)^2}{R}$$

$$= I^2 R$$

use Ohm's law

(d) Now let the terminals discharge after being charged up to potential difference V_0.

Ohm's law: $V = V_2 - V_1 = IR$

$$\frac{dQ_1}{dt} = -\frac{dQ}{dt} = -C\frac{dV}{dt}$$

$$\therefore \quad \frac{dV}{dt} = -\underbrace{\frac{1}{RC}}_{\equiv 1/\tau} V \qquad \tau = RC = \epsilon/\sigma$$

$$\boxed{\frac{dV}{dt} = -\frac{V}{\tau}}$$

Integrate: $\frac{dV}{V} = -\frac{dt}{\tau} \Rightarrow \ln\frac{V}{V_0} = -\frac{t}{\tau}$

$$\boxed{V = V_0 e^{-t/\tau}}$$

A good conductor (like copper) has

$$\epsilon = \epsilon_0 = 8.85 \times 10^{-12}\ F/m$$

$$\sigma = 1/\rho = \frac{1}{1.7\times 10^{-8}}\ \frac{1}{\Omega\text{-}m}$$

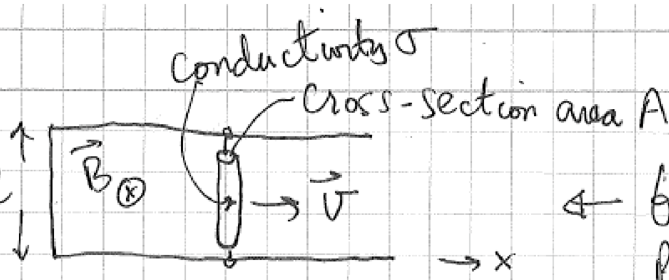
$$\Longrightarrow \tau = \frac{\epsilon}{\sigma} = (8.85\times 10^{-12})(1.7\times 10^{-8}) \simeq 10^{-20}\ s$$

You might think this is a problem, because even for very small distances between the terminals, say 10^{-6} m, it corresponds to a velocity

$$\frac{10^{-6}\ m}{10^{-20}\ s} = 10^{14}\ m\text{-}s^{-1} \gg c,$$

much greater than the speed of light. This isn't really a problem because individual charges move only a tiny distance to neutralize the terminals. The real problem is that this time is much shorter than a typical collision time, $\tau_c \sim 3\times 10^{-14}$ s, in a good conductor. This means Ohm's law isn't really valid on a time scale as short as $\tau \sim 10^{-20}$ s.

4)



← flux <u>increases</u> into page as rod moves to right

a) $\frac{d\Phi_B}{dt} = -\mathcal{E} = -V = -IR$

$$\frac{d}{dt}\int \vec{B}\cdot d\vec{a} = Bl\frac{dx}{dt} = Blv = -IR$$

↑ into page

$$\Rightarrow \boxed{I = -Blv/R}$$

direction <u>opposite</u> that given right hand rule → counter-clockwise

Here, R is resistance:

$$R = l/\sigma A$$

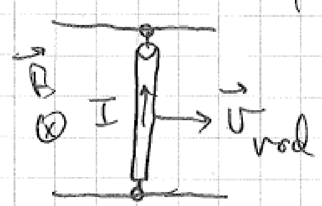
So $\boxed{I = Bv\sigma A}$ ↻ in direction

↺ Induced I

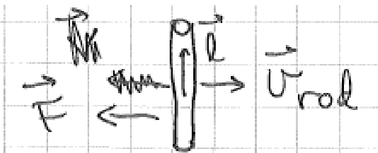
b) What is $\vec{F}$ on rod? From Lorentz Law:

$$\vec{F} = q\vec{v}\times\vec{B}$$

velocity of charge carrier <u>NOT</u> rod! Given by direction of <u>induced</u> current



Direction of + charges moving in rod is ↑, so $q\vec{v}\times\vec{B}$ is to the <u>left</u>, opposing the initial vel. of rod to the right



To get the magnitude of the $\vec{F}$ we use

$\vec{F} = I\vec{l}\times\vec{B}$ with $\vec{l}$ in direction of I

$|\vec{F}| = IlB = B^2 l v_{rod} \sigma A$

c) If rod is given an initial velocity $\vec{v}_0$ to the right, what is $\vec{v}(t)$ later on?

$$\vec{F} = m\frac{d\vec{v}}{dt} = -B^2 l \vec{v}\sigma A$$

$\uparrow$ $\vec{F}$ opposes $\vec{v}$

$m = \rho_m A l$, mass of rod

$$\frac{dv}{v} = -\frac{B^2 l \sigma A}{m} = -\frac{B^2\sigma}{\rho_m}$$

integrating $\int_{v_0}^{v(t)} (\ ) = -\int_0^t$

$$\Rightarrow v(t) = v_0 e^{-B^2\sigma t/\rho_m}$$

$$\boxed{v(t) = v_0 e^{-\Gamma t}, \quad \Gamma \equiv \frac{B^2\sigma}{\rho_m}}$$

d) Initial K.E. $= \frac{1}{2}mv_0^2$; $t\to\infty$ all of this is lost.

Where does it go? Has to go into resistor as heat: $P = I^2R = VI$. From part a):

$\cancel{v_0 = I}$ $I = Bv\sigma A$, which is a funct. of time

$I(t) = B\sigma A v(t)$

$\Rightarrow P = I^2R = B^2\sigma^2A^2R\, v^2(t)$

Finally, $(\cancel{B^2}\sigma^{\cancel{2}} A^2 R)\dfrac{v_0^2 \rho_m}{2\cancel{B^2}\cancel{\sigma}}$ $\leftarrow \dfrac{m}{\ell A}$

$$= \sigma A^{\cancel{2}} R v_0^2 \frac{m}{2\ell \cancel{A}} = \underbrace{\frac{\sigma A R}{\ell}}\left(\frac{1}{2} m v_0^2\right)$$

$$\frac{\sigma A}{\ell}\cdot\frac{\ell}{\sigma A} = 1$$

Energy dissipated $= \frac{1}{2} m v_0^2$! ✓