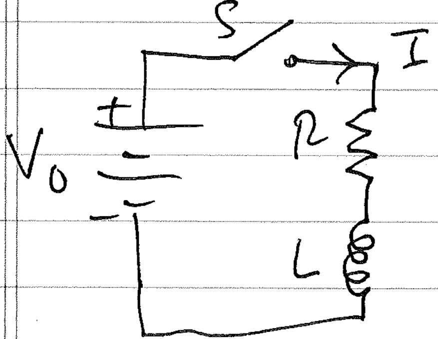
$\boxed{\text{Circuits}}$ RC, RL, LC, RLC ...

Kirchoff's Law : $\sum_{\substack{\text{closed} \\ \text{circuit}}} V = 0$ } From $\oint \vec{E} \cdot \vec{dl} = 0$

$\Uparrow$

Quasi-static Approx!

E.g: R-L circuit



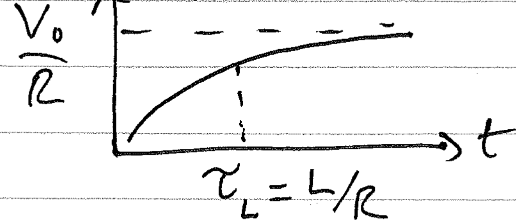
$$\sum_i V = 0 \rightarrow V_0 = \underbrace{V_R}_{IR} + \underbrace{V_L}_{L\frac{dI}{dt}}$$

$$V_0 = IR + L\,dI/dt$$

For t near zero, I is small so: $\frac{dI}{dt} \approx \frac{V_0}{L}$

$$\Rightarrow I \approx \frac{V_0}{L} t \quad (I(t=0) = 0)$$

For large t, $\frac{dI}{dt} \rightarrow 0$, so standard solution of battery + R : $I \rightarrow V_0/R$



Full solution:

$$I(t) = \underbrace{A e^{-\frac{R}{L}t}}_{\text{homogeneous soln}} + \underbrace{V_0/R}_{\text{particular soln}}$$

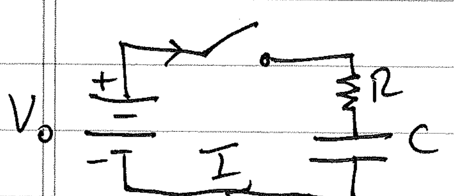
Init. conditions: $I(0) = 0$ gives $A = -\frac{V_0}{R}$

So, $\boxed{I(t) = \frac{V_0}{R}\left(1 - e^{-Rt/L}\right)}$

Inductors act like ∞ resistance when $\frac{dI}{dt}$ is large. But they act like short circuits for $I \approx$ constant or $\frac{dI}{dt} \approx 0$

R-C circuits

Review these from P161!



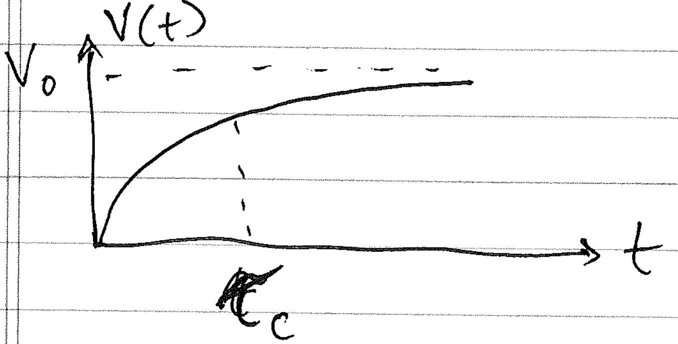
$$V_0 = V_R + V_C = IR + V_C$$

$$I = C\frac{dV_C}{dt}$$

$$\rightarrow V_0 = RC\frac{dV_C}{dt} + V_C$$

or $$\boxed{\frac{dV_C}{dt} = -\frac{1}{RC}V_C + \frac{V_0}{RC}}$$

Full soln : $V(t) = V_0 \left(1 - e^{-t/RC}\right)$



$\tau_c \equiv RC$ time-const.

When $V(t) \to V_0$ (stops changing), I stops:

* Caps. act as open circuits or ∞ resistors when dV/dt is small

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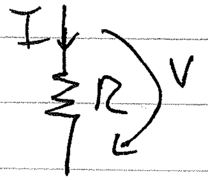
AC circuits Review complex vars, exponentials ~~notation~~, etc

$$e^{-i\omega t} = \cos \omega t - i \sin \omega t$$

Impedance

Using complex notation: $V(t) = \mathrm{Re}(\tilde{V} e^{-i\omega t})$
Same with $I(t)$.

$\rightarrow$ Impedance $Z = \tilde{V} / \tilde{I}$

Examples: 

$V = IR$, $\boxed{Z_R = \frac{V}{I} = R}$, real

Using complex notation we write

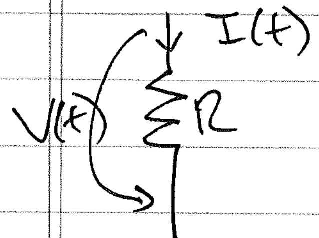
$$V(t) = \hat{V} e^{-i\omega t}$$

$$I(t) = \hat{I} e^{-i\omega t}$$

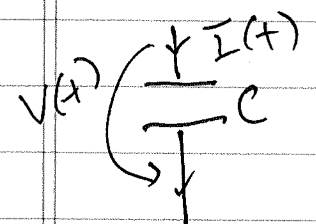
with $\hat{V}$, $\hat{I}$ the complex (in general) phasors.

<u>Impedance Z</u>. This quantity is defined as the ration of the complex V/I across a circuit element

<u>Examples</u>

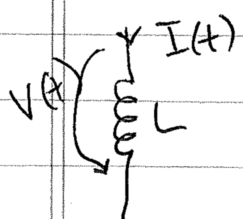


$$\boxed{Z_R = \frac{V}{I} = \frac{IR}{I} = R}$$, a real # here



$$Z_C = \frac{V}{I} \; ; \; Q = CV, \; I = \frac{dQ}{dt} = C\frac{dV}{dt}$$

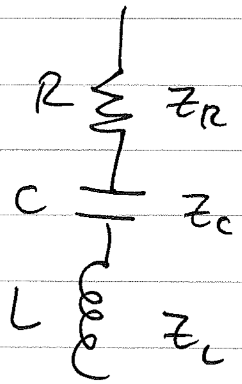
$$\rightarrow I = C(-i\omega)V, \text{ so } \boxed{Z_C = \frac{1}{-i\omega C}}$$



$$-\mathcal{E} = V = L\frac{dI}{dt} = L(-i\omega)I$$

$$\rightarrow \boxed{Z_L = \frac{V}{I} = -i\omega L}$$

Impedance of circuits:

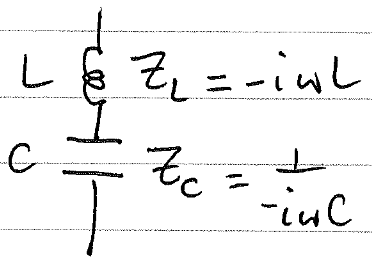


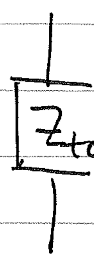
In series they effective Z of circuit is $\sum_i Z$'s

$$\rightarrow Z = Z_R + Z_L + Z_C$$

$$\boxed{Z = R + \frac{1}{-i\omega C} - i\omega L}$$

For following circuit:



$\Rightarrow$ 

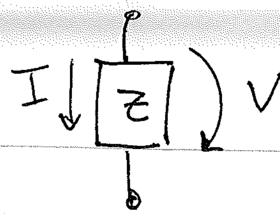
$$Z_{total} = -i\omega L + \frac{1}{-i\omega C} = \frac{1-\omega^2 LC}{-i\omega C}$$

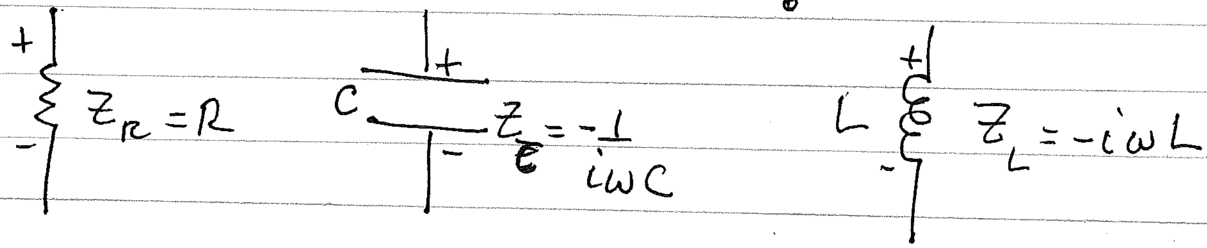
Here, as we'll see in a bit, when

$$\omega = \omega_0 = \frac{1}{\sqrt{LC}}, \quad Z_{total} = 0$$

$\Rightarrow$ Out of phase impedances cancel!

Summary : AC circuit





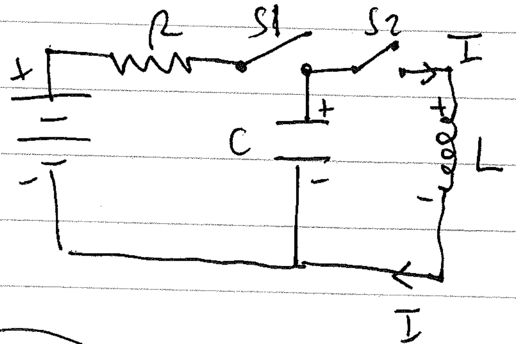
$$Z_R = R \qquad Z_C = \frac{-1}{i\omega C} \qquad Z_L = -i\omega L$$

DC ? limit $\omega \to 0$:

$Z_R = R$; $Z_C \to \infty$ an open circuit! ; $Z_L \to 0$ a short circuit!

Note that this components are called <u>passive</u> : they <u>respond</u> to external conditions, e.g. changes in V or I

Back to LC circuit :



- close S1 till C is charged
- open (close) S1 (S2)

$Q = CV_c$; $I_c = -dQ/dt$ Why ?

$I_c = -C\dfrac{dV_c}{dt}$

induced

$$V_L = V_c \; ; \; I_L = +I_c = -C\frac{dV_c}{dt} = -C\frac{dV_L}{dt}$$

also $\boxed{V_L = L\dfrac{dI_L}{dt} = -LC\dfrac{d^2V_L}{dt^2}}$

or SHO : $$\boxed{\frac{d^2V_L}{dt^2} + \frac{1}{LC} V_L = 0}$$

$$= \boxed{\omega_0^2 \equiv \frac{1}{LC}}$$ resonant freq.

Solution : (P160)

$V_L(t) = a\cos\omega_0 t + b\sin\omega_0 t$

B.C.'s $I_L(0) = 0$, $V_L(0) = V_0$

$\rightarrow$ $a = V_0$, $b = 0$

$I_L(t) = -C\frac{dV_L}{dt}$

$= \frac{a}{\sqrt{L/C}} \sin\omega_0 t$

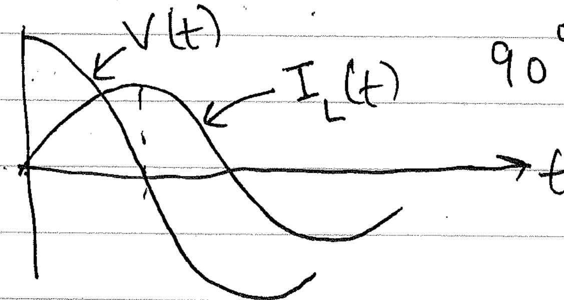
$\frac{-b}{\sqrt{L/C}} \cos\omega t$

$(V_c =)$ $V_L(t) = V_0 \cos\omega_0 t$

$I_L(t) = I_0 \sin\omega_0 t$

90° out of phase

Ratio of $\frac{V_0}{I_0} \equiv Z = \sqrt{L/C}$ characteristic impedance



90° out of phase!

(Energy)

In Cap:

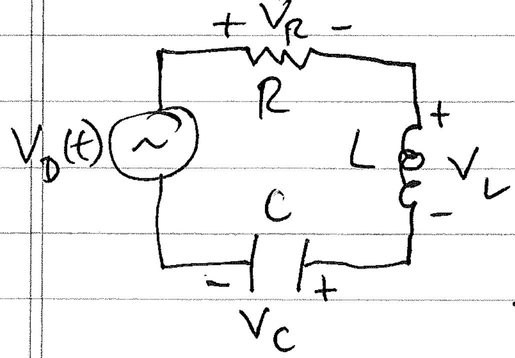
$U_c = \frac{1}{2} C V_c^2 = \frac{1}{2} C V_0^2 \cos^2\omega_0 t$

$\underbrace{}_{LI_0^2}$!

In Inductor

$U_L = \frac{1}{2} L I_L^2 = \frac{1}{2} L I_0^2 \sin^2\omega_0 t$

RLC circuit



$$V_D(t) = V_R + V_L + V_C$$

$$I_R = I_L = I_C \equiv I(t)$$

$$I(t) = \underbrace{I_{\text{Homog.}}}_{\text{transient}} + \underbrace{I_{\text{particular}}}_{\text{steady-state}}$$

w/ $V_D = 0$
after C is charged

Transient solution: $V_D = 0$

$$V_R + V_L + V_C = 0 = IR + L\frac{dI}{dt} + \frac{Q}{C}$$

$$R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{I}{C} = 0$$

Damped S.H.O.

$$\ddot{I} + \Gamma\dot{I} + \omega_0^2 I = 0$$

Damping

where $\omega_0 \equiv 1/\sqrt{LC}$ and $\Gamma = L/R$

Ansatz: $I = \mathrm{Re}[\tilde{I} e^{st}]$ where s is complex in general

subing above gives:

$$\underbrace{(s^2 + s\Gamma + \omega_0^2)}_{=0}\tilde{I} = 0 \Rightarrow \boxed{s_\pm = -\frac{\Gamma}{2} \pm \frac{\sqrt{\Gamma^2 - 4\omega_0^2}}{2}}$$

Two regimes: underdamped and overdamped

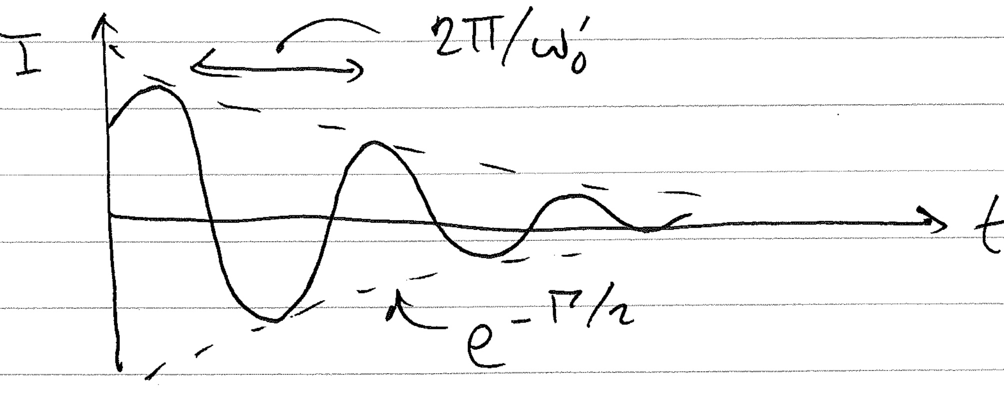
(1) Underdamped: $\omega_0 > \Gamma/2$

$$\Rightarrow s_\pm = \pm i\omega_0' - \Gamma/2$$

$$\omega_0' = \sqrt{\omega_0^2 - \Gamma^2/4}$$

Then, $I(t) = \mathrm{Re}\left(\tilde{I}\underbrace{e^{-\Gamma/2}}_{\text{dampin}}\underbrace{e^{\pm i\omega_0' t}}_{\text{oscillations}}\right)$

$$\boxed{I(t) = A e^{-\Gamma/2}\cos(\omega_0' t - \phi)}$$



Next, consider steady-state solution for when $V_D(t) = \mathrm{Re}[V_0 e^{-i\omega t}]$

$$\rightarrow I(t) = \mathrm{Re}[\tilde{I} e^{-i\omega t}]$$

$$Z_{total} = Z_R + Z_C + Z_L = R + i\left(\frac{1}{\omega C} - \omega L\right)$$

$\Rightarrow$ ~~$I(t) = \mathrm{Re}\left[V_0 e^{-i\omega t} \right.$~~

Now, $Z_{total} = \dfrac{V_0}{I} = \dfrac{V_0}{\tilde{I}}$

or $I = \dfrac{V}{Z_{total}}$, taking Real part:

$$I = \mathrm{Re}\left[\frac{V_0 e^{-i\omega t}}{Z_{total}} \right]$$

$$I(t) = \mathrm{Re}\left[\frac{V_0 e^{-i\omega t}}{R + i\left(\frac{1}{\omega C} - \omega L\right)} \right] = \frac{V_0}{|Z_{tot}|} \cos(\omega t - \phi)$$

where $|Z_{total}| = \sqrt{R^2 + \frac{1}{\omega^2 C^2}\left(1 - \omega^2/\omega_0^2\right)}$

$$\phi = \tan^{-1}\left(\frac{1}{\omega R C} - \frac{\omega L}{R} \right)$$

<u>ANOTHER way</u> to express this is in terms of <u>both</u> $\sin(\omega t)$ <u>and</u> $\cos(\omega t)$! After all, $\cos(\omega t - \phi) \rightarrow$ broken into sin & cos terms.

$$I(t) = \mathrm{Re}\left[\frac{V_0 e^{-i\omega t}}{Z_{tot}}\right]$$

$$= \mathrm{Re}\left[\frac{V_0 Z^* e^{-i\omega t}}{\underbrace{Z Z^*}_{\substack{\text{real} \\ = |Z|^2}}}\right] = \frac{V_0}{|Z|^2}\,\mathrm{Re}\left[Z^* e^{-i\omega t}\right]$$

$$= \frac{V_0}{R^2 + \frac{1}{\omega^2 C^2}(1-\omega^2/\omega_0^2)}\;\underbrace{\mathrm{Re}\left[\left(R - i\left(\tfrac{1}{\omega C} - \omega L\right)\right)(\cos\omega t - i\sin\omega t)\right]}_{R\cos\omega t - \left(\frac{1}{\omega C} - \omega L\right)\sin\omega t}$$

$$= \frac{V_0\left[R\cos\omega t - \left(\frac{1}{\omega C} - \omega L\right)\sin\omega t\right]}{R^2 + \frac{\omega_0^2 - \omega^2}{\omega_0^2\omega^2 C^2}}$$

Define the <u>response</u> <u>function</u>:

$$F(\omega) \equiv \frac{V_c}{V_0} \leftarrow \text{voltage across cap.}$$

$$V_c = I Z_c = \frac{V_0}{Z_{tot}} \cdot Z_c$$

$$\rightarrow \quad F(\omega) = \frac{\frac{i}{\omega C}}{R + i\left(\frac{1}{\omega C} - \omega L\right)} = \frac{1}{-i\omega RC - \omega^2 LC + 1}$$

$$\boxed{F(\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}}$$

Resonance when $F(\omega)$ is max., @ $\omega = \omega_0$

Here $F(\omega) \to \infty$ if no damping: $\Gamma \to 0$

If low damping $\Gamma \ll \omega_0$, $F(\omega)$ will be peaked at $\omega \approx \omega_0$. Let's see this:

$$\Delta \equiv \omega - \omega_0, \quad \Gamma, \Delta \ll \omega, \omega_0$$

Write

$$\omega_0^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega)$$

$$= -\Delta(\Delta + 2\omega_0)$$

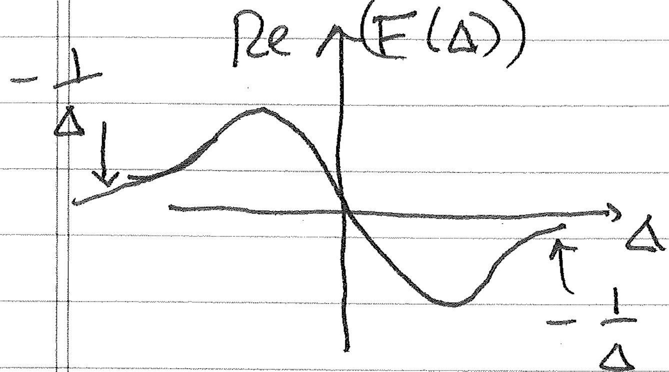
$$\approx -2\omega_0 \Delta$$

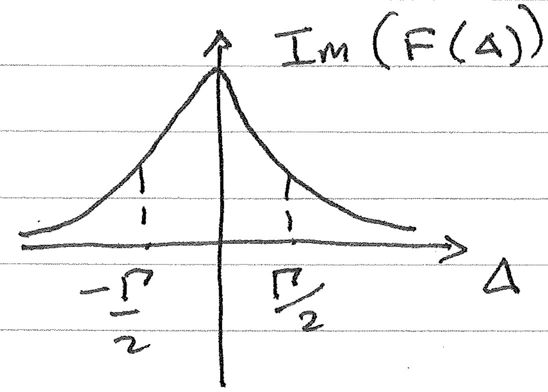
So,

$$\boxed{F(\Delta) \approx \frac{\omega_0/2}{-\Delta - i\Gamma/2}}$$

$$\mathrm{Re}\left(F(\Delta)\right) = \frac{\omega_0}{2}\left(\frac{-\Delta}{\Delta^2 + \Gamma^2/4}\right)$$

$$\mathrm{Im}\left(F(\Delta)\right) = \frac{\omega_0}{2}\left(\frac{\Gamma/4}{\Delta^2 + \Gamma^2/4}\right)$$





This fn is called a "Lorentzian" with $\Gamma \equiv$ full width at half max.

$$\rightarrow V_{cap}(t) = \mathrm{Re}\left[F(\omega) V_0 e^{-i\omega t}\right]$$

$$V_{cap}(t) = \underbrace{\mathrm{Re}\left(F(\omega)\right) V_0 \cos\omega t}_{\text{in phase with drive volt. } V_D(t)} + \underbrace{\mathrm{Im}\left(F(\omega)\right) V_0 \sin\omega t}_{\text{out of phase ("in quadrature") with } V_D(t)}$$

$$\boxed{= \frac{\omega_0/2}{\sqrt{\Delta^2 + \Gamma^2/4}}\, V_0 \cos\left(\omega t - \arg(F(\omega))\right)}$$

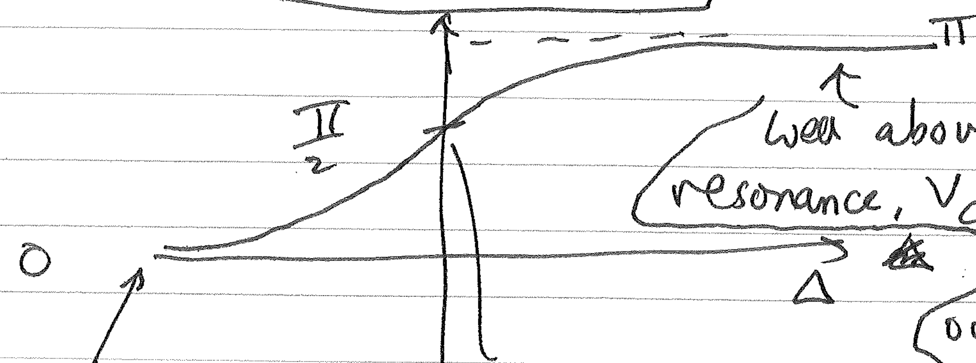
→ Amplitude of $V_{cap}(t)$ is <u>max</u>

when resonance, $\Delta = 0$

→ There's a phase shift:

$$\phi(\omega) = \arg(F(\omega)) = -\arg(Z_{tot})$$

$$\boxed{\phi(\omega) = -\tan^{-1}\left(\Gamma/2\Delta\right)}$$



well above resonance, V_C & V_D 180° out of phase

well below resonance, V_{cap} & V_D <u>in</u> phase

at resonance, V_{cap} 90° out of phase w/ V_D