Separation of Variables

We now have an equation that provides us with a means to get the wave functions, which, in turn, provide us with the means to extract the dynamic quantities of interest. Remember, that Schrödinger’s equation is in quantum mechanics what \( F = ma \) is in classical mechanics. If you solve Newton’s first law, knowing the potential acting on a particle, you can get a description of the behavior of that particle, \( x(t) \). So, how do we extract \( \Psi \) from Schrödinger’s equation:

\[
\text{i}\hbar \frac{\partial \Psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + V(x, t)\Psi(x, t) \tag{1}
\]

We notice first that this equation is a partial differential equation, consisting of terms with derivatives in time, \( t \) and position, \( x \). The standard method of solving such an equation is the method of separation of variables in which we search for a solution for \( \Psi(x, t) \) that is a product of two functions, each of which is a function of only one variable:

\[
\Psi(x, t) = \psi(x)\phi(t)
\]

putting this into Schrödinger’s equation yields:

\[
\text{i}\hbar \frac{d\phi(t)}{dt} \psi(x) = -\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} \phi(t) + V(x, t)\psi(x)\phi(t)
\]

and dividing by \( \Psi \):

\[
\text{i}\hbar \frac{d\phi(t)}{dt} \frac{1}{\phi(t)} = -\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} \frac{1}{\psi(x)} + V(x, t)
\]

Now, on the left hand side, we have only functions of time, and on the right hand side, we have only functions of position – except for the potential \( V \)! To overcome this obstacle, let’s do what physicists do best and only consider potentials that do not explicitly depend on time, i.e., \( V(x, t) = V(x) \).

\[
\text{i}\hbar \frac{d\phi(t)}{dt} \frac{1}{\phi(t)} = -\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} \frac{1}{\psi(x)} + V(x)
\]

OK, now that each side depends only on one variable, and these are independent, then it must be true that each side equals a constant:
\[ i\hbar \frac{d\phi(t)}{dt} \frac{1}{\phi(t)} = E \]
\[ -\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} \frac{1}{\psi(x)} + V(x) = E \]

I label the constant E with some feeling for the individual operators that we used for the development of the Schrödinger's equation. Now we have two ordinary differential equations that we can more easily solve:

\[ i\hbar \frac{d\phi(t)}{dt} = E\phi(t) \]
\[ -\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \]

The second of these two equations is known as the time-independent Schrödinger's equation. Before we attack its solutions, let's look at the solutions to the time-dependent part:

\[ i\hbar \frac{d\phi(t)}{dt} = E\phi(t) \Rightarrow \]
\[ \frac{d\phi(t)}{\phi(t)} = -\frac{iE}{\hbar} dt \Rightarrow \]
\[ \int \frac{d\phi(t)}{\phi(t)} = \int -\frac{iE}{\hbar} dt \Rightarrow \]
\[ \log \phi(t) = -\frac{iE}{\hbar} t + C \Rightarrow \]
\[ \phi(t) = e^{-\frac{iE}{\hbar} t + C} = e^{-\frac{iE}{\hbar} t} e^C = Ce^{-\frac{iE}{\hbar} t} \]

Since we are eventually looking for solutions to \( \Psi = \phi\psi \), we will incorporate the constant C into \( \psi \) and give the solution \( \phi \) as:

\[ \phi(t) = e^{-\frac{iE}{\hbar} t} \]

so that the general solution to the time-dependent wave equation is:

\[ \Psi(x, t) = \psi(x)\phi(t) = \psi(x)e^{-\frac{iE}{\hbar} t} \]

**Homework:** Show that \( E \) must exceed the minimum value of \( V \) in order that there are normalizable solutions \( \Psi \). **Hint:** Consider the relative sign of \( \Psi \) and its second derivative, and the impact this has on its ability to be normalized.