Free Particle

So let's begin our exploration of solutions to the Time-Independent Schrödinger's equation with the simplest situation. Let the time independent potential be a constant with respect to position. Then, we know that the value of the constant has no impact on the behavior of the particle, so we set the value to zero, V(x) = V = 0. Then, the TISE becomes:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

The general solution to this differential equation (which can be found by integrating twice) is:

$$\psi(x) = Ae^{ikx} + Be^{-ikx},\ k \equiv \frac{\sqrt{2mE}}{\hbar}$$

and we recover the free particle wave function that we postulated earlier:

$$\Psi(x,t) = \psi(x)\varphi(t) = \left(Ae^{ikx} + Be^{-ikx}\right)e^{-i\omega t} = Ae^{i(kx-\omega t)} + Be^{-i(kx-\omega t)},\ \omega = \frac{E}{\hbar}$$

Here, k can take on any positive value (depending upon the value of E). We can let k take on both positive and negative values, where k > 0 corresponds to waves traveling to the right, and k < 0, to the left. We then have:

$$\Psi(x,t) = Ae^{i(kx-\omega t)}$$

as before, or, in terms of k alone:

$$\Psi_k(x,t) = Ae^{i\left(kx-\frac{\hbar k^2}{2m}t\right)}$$

Now, this wave function should really bother you. It doesn't? OK, then, normalize it:

$$\int_{-\infty}^{\infty}\Psi_k^*(x,t)\Psi_k(x,t)dx = \int_{-\infty}^{\infty}A^2e^{i\left(kx-\frac{\hbar k^2}{2m}t\right)}e^{-i\left(kx-\frac{\hbar k^2}{2m}t\right)}dx$$
$$= A^2\int_{-\infty}^{\infty}dx = A^2(\infty)$$

So that these wave functions cannot represent real particles, at least with single values of the wave number k, since the probability to find the particle extends to plus/minus infinity. We can address this by remembering that the general solution is a linear combination of the separable solutions. Then, as before, we can add wave functions with a distribution of wave numbers together to form a wave packet, such that the total wave function vanishes at infinity and is therefore normalizable. Then, as before, we have for the general solution:

$$\Psi(x, t) = \int_{-\infty}^{\infty} \phi(k)\Psi_k(x, t)dk,$$

where $\phi(k)$ now represents the constants c_n that we saw in the general solution, but is a continuous set, rather than having discreet indices so we integrate rather than sum. We find the set of constants in the usual way:

$$\Psi(x,0) = \int_{-\infty}^{\infty} \phi(k)\psi_k(x)dk,$$

and the $\phi(k)$ can be found by the inverse Fourier transform.