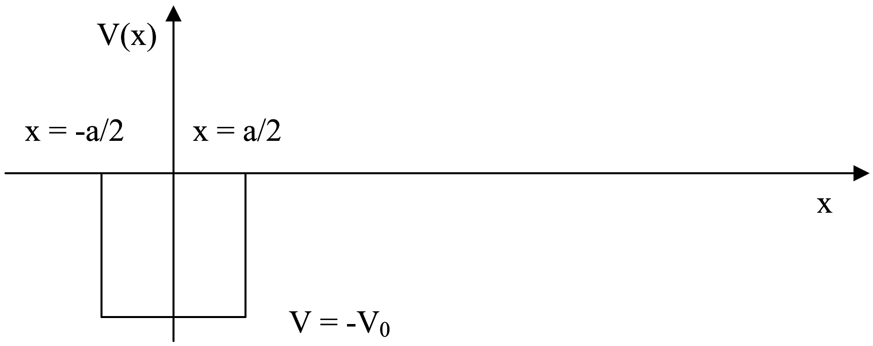
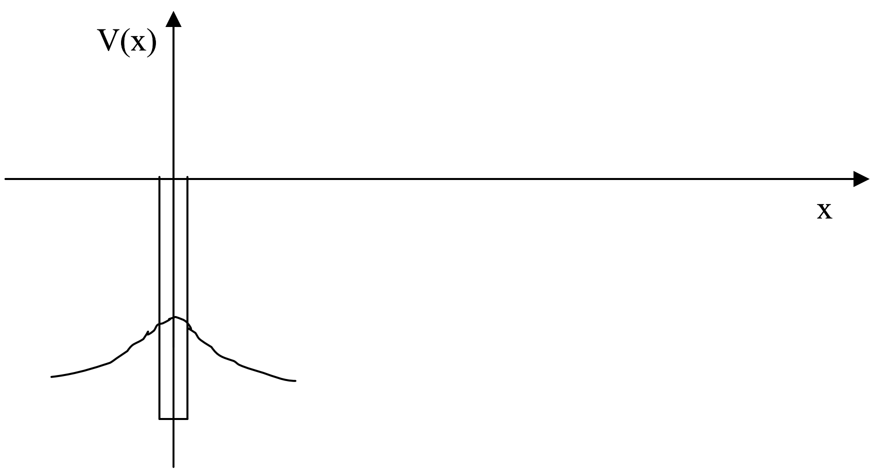
The Delta-Function Potential

As our last example of one-dimensional bound-state solutions, let us re-examine the finite potential well:



and take the limit as the width, a, goes to zero, while the depth, V_0, goes to infinity keeping their product aV_0 to be constant, say U_0. In that limit, then, the potential becomes:

$$V(x) = -U_0\delta(x)$$



and we can have a sense that if there is at least one bound state of this potential, it should look (as drawn above) like the ground state of the finite square well. Let's examine the TISE for this potential:

$$\hat{H}\psi = E\psi \Rightarrow$$

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - U_0\delta(x)\right]\psi = E\psi \Rightarrow$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - U_0\delta(x)\psi = E\psi \qquad \text{at x} = 0$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi \qquad \text{otherwise}$$

Let's first take a look at the region outside of x = 0. Here, we can rewrite the TISE as:

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\psi \Rightarrow$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{-2mE}{\hbar^2}\psi = \kappa^2\psi$$

$$\kappa = \frac{\sqrt{-2mE}}{\hbar}$$

Here, kappa is real since the energy is negative. This simple differential equation has solutions we know:

$$\psi(x) = Ae^{-\kappa x} + Be^{\kappa x}$$

For negative x (left side of the potential), ψ will blow up as x goes to negative infinity, so A must be zero. On the right side the same thing happens and there, the constant B must be zero. So, we have:

$$\psi(x) = \begin{cases} Be^{\kappa x} & x < 0 \\ Ae^{-\kappa x} & x > 0 \end{cases}$$

At x = 0, the wavefunction must be continuous, so A = B and we have:

$$\psi(x) = \begin{cases} Ae^{\kappa x} & x \le 0 \\ Ae^{-\kappa x} & x \ge 0 \end{cases}$$

Now, we must use the information at x = 0. In the region near x = 0, the TISE gives us:

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - U_0\delta(x)\psi = E\psi$$

we integrate both sides with respect to x over an infinitesimally small region around the delta potential, say from –ε to +ε:

$$\int_{-\varepsilon}^{\varepsilon} \frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2}\, dx - \int_{-\varepsilon}^{\varepsilon} U_0 \delta(x)\psi(x)\, dx = \int_{-\varepsilon}^{\varepsilon} E\psi(x)\, dx \Rightarrow$$

$$\frac{-\hbar^2}{2m}\left(\left.\frac{\partial \psi(x)}{\partial x}\right|_{\varepsilon} - \left.\frac{\partial \psi(x)}{\partial x}\right|_{-\varepsilon} \right) - U_0\psi(0) = E\int_{-\varepsilon}^{\varepsilon} \psi(x)\, dx$$

Now, let's let ε go to zero. Then the right hand side of the equation will go to zero since ψ is finite and we integrate it over a zero width. This gives us:

$$\left(\left.\frac{\partial \psi(x)}{\partial x}\right|_{\varepsilon} - \left.\frac{\partial \psi(x)}{\partial x}\right|_{-\varepsilon} \right) = \frac{-2mU_0}{\hbar^2}\psi(0)$$

Now, from above (and letting epsilon go to zero):

$$\text{for } x > 0,\ \frac{\partial \psi(x)}{\partial x} = -A\kappa e^{-\kappa x} \Rightarrow$$

$$\left.\frac{\partial \psi(x)}{\partial x}\right|_{0} = -A\kappa$$

$$\text{for } x < 0,\ \frac{\partial \psi(x)}{\partial x} = A\kappa e^{\kappa x} \Rightarrow$$

$$\left.\frac{\partial \psi(x)}{\partial x}\right|_{-0} = A\kappa$$

So that

$$\left(\left.\frac{\partial \psi(x)}{\partial x}\right|_{\varepsilon} - \left.\frac{\partial \psi(x)}{\partial x}\right|_{-\varepsilon} \right) = -A\kappa - A\kappa = -2A\kappa = \frac{-2mU_0}{\hbar^2}\psi(0)$$

Now, ψ(0) = A, so:

$$-2A\kappa = \frac{-2mU_0}{\hbar^2} A \Rightarrow$$

$$\kappa = \frac{mU_0}{\hbar^2}$$

So that there is one, and only one allowed energy level:

$$\kappa = \frac{mU_0}{\hbar^2} = \frac{\sqrt{-2mE}}{\hbar} \Rightarrow$$

$$E = -\frac{mU_0^2}{2\hbar^2}$$

To be complete, we can find the constant A via the normalization condition:

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx \; = \; 2|A|^2 \int_{0}^{\infty} e^{-2\kappa x} dx \; = \; \frac{|A|^2}{\kappa} \; = \; 1 \; \Rightarrow$$

$$A \; = \; \sqrt{\kappa} \; = \; \frac{\sqrt{mU_0}}{\hbar}$$

Then there is one and only one bound state, and one energy eigenvalue:

$$\psi(x) \; = \; \frac{\sqrt{mU_0}}{\hbar} \, e^{-\frac{mU_0}{\hbar^2}|x|}, \qquad E \; = \; -\frac{mU_0^2}{2\hbar^2}$$