4.4 SPIN

In classical mechanics, a rigid object admits two kinds of angular momentum: orbital (\(L = r \times p\)), associated with the motion of the center of mass, and spin (\(S = I \omega\)), associated with motion about the center of mass. For example, the earth has orbital angular momentum attributable to its annual revolution around the sun, and spin angular momentum coming from its daily rotation about the north-south axis. In the classical context this distinction is largely a matter of convenience, for when you come right down to it, \(S\) is nothing but the sum total of the “orbital” angular momenta of all the rocks and dirt clods that go to make up the earth, as they circle around the axis. But an analogous thing happens in quantum mechanics, and here the distinction is absolutely fundamental. In addition to orbital angular momentum, associated (in the case of hydrogen) with the motion of the electron around the nucleus (and described by the spherical harmonics), the electron also carries another form of angular momentum, which has nothing to do with motion in space (and which is not, therefore, described by any function of the position variables \(r, \theta, \phi\) but which is somewhat analogous to classical spin (and for which, therefore, we use the same word). It doesn’t pay to press this analogy too far: The electron (as far as we know) is a structureless point particle, and its spin angular momentum cannot be decomposed into orbital angular momenta of constituent parts (see Problem 4.25).\(^{25}\)

Suffice it to say that elementary particles carry intrinsic angular momentum (\(S\)) in addition to their “extrinsic” angular momentum (\(L\)).

The algebraic theory of spin is a carbon copy of the theory of orbital angular momentum, beginning with the fundamental commutation relations:\(^{26}\)

\[
[S_x, S_y] = i\hbar S_z, \quad [S_y, S_z] = i\hbar S_x, \quad [S_z, S_x] = i\hbar S_y.
\]  

It follows (as before) that the eigenvectors of \(S^2\) and \(S_z\) satisfy\(^{27}\)

\[
S^2|s m\rangle = \hbar^2 s(s + 1)|s m\rangle; \quad S_z|s m\rangle = \hbar m|s m\rangle;
\]  


\(^{26}\)We shall take these as postulates for the theory of spin; the analogous formulas for orbital angular momentum (Equation 4.99) were derived from the known form of the operators (Equation 4.96). In a more sophisticated treatment they can both be obtained from rotational invariance in three dimensions (see, for example, Leslie E. Ballentine, Quantum Mechanics: A Modern Development, World Scientific, Singapore (1998), Section 3.3). Indeed, these fundamental commutation relations apply to all forms of angular momentum, whether spin, orbital, or the combined angular momentum of a composite system, which could include some spin and some orbital.

\(^{27}\)Because the eigenstates of spin are not functions, I will use the “ket” notation for them. (I could have done the same in Section 4.3, writing \(|l m\rangle\) in place of \(Y^l_m\), but in that context the function notation seems more natural.) By the way, I’m running out of letters, so I’ll use \(m\) for the eigenvalue of \(S_z\), just as I did for \(L_z\) (some authors write \(m_l\) and \(m_s\) at this stage, just to be absolutely clear).
and

\[ S_{\pm} |s m\rangle = \hbar \sqrt{s(s + 1) - m(m \pm 1)} |s (m \pm 1)\rangle, \quad [4.136] \]

where \( S_{\pm} = S_x \pm iS_y \). But this time the eigenvectors are not spherical harmonics (they're not functions of \( \theta \) and \( \phi \) at all), and there is no \textit{a priori} reason to exclude the half-integer values of \( s \) and \( m \):

\[ s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots; \quad m = -s, -s + 1, \ldots, s - 1, s. \quad [4.137] \]

It so happens that every elementary particle has a \textit{specific and immutable} value of \( s \), which we call the \textit{spin} of that particular species: pi mesons have spin 0; electrons have spin 1/2; photons have spin 1; deltas have spin 3/2; gravitons have spin 2; and so on. By contrast, the \textit{orbital} angular momentum quantum number \( l \) (for an electron in a hydrogen atom, say) can take on any (integer) value you please, and will change from one to another when the system is perturbed. But \( s \) is \textit{fixed}, for any given particle, and this makes the theory of spin comparatively simple.\(^{28}\)

---

**Problem 4.25** If the electron were a classical solid sphere, with radius

\[ r_e = \frac{e^2}{4\pi \epsilon_0 mc^2} \quad [4.138] \]

(the so-called \textit{classical electron radius}, obtained by assuming the electron's mass is attributable to energy stored in its electric field, via the Einstein formula \( E = mc^2 \)), and its angular momentum is \((1/2)\hbar\), then how fast (in m/s) would a point on the "equator" be moving? Does this model make sense? (Actually, the radius of the electron is known experimentally to be much less than \( r_e \), but this only makes matters worse.)

---

\(^{28}\)Indeed, in a mathematical sense, spin 1/2 is the simplest possible nontrivial quantum system, for it admits just two basis states. In place of an infinite-dimensional Hilbert space, with all its subtleties and complications, we find ourselves working in an ordinary 2-dimensional vector space; in place of unfamiliar differential equations and fancy functions, we are confronted with \( 2 \times 2 \) matrices and 2-component vectors. For this reason, some authors begin quantum mechanics with the study of spin. (An outstanding example is John S. Townsend, \textit{A Modern Approach to Quantum Mechanics}, University Books, Sausalito, CA, 2000.) But the price of mathematical simplicity is conceptual abstraction, and I prefer not to do it that way.
4.4.1 Spin 1/2

By far the most important case is $s = 1/2$, for this is the spin of the particles that make up ordinary matter (protons, neutrons, and electrons), as well as all quarks and all leptons. Moreover, once you understand spin 1/2, it is a simple matter to work out the formalism for any higher spin. There are just two eigenstates: $|\frac{1}{2}\ 1\rangle$ and $|\frac{1}{2} -\frac{1}{2}\rangle$, which we call spin up (informally, $\uparrow$), and $|\frac{1}{2} -\frac{1}{2}\rangle$, which we call spin down ($\downarrow$). Using these as basis vectors, the general state of a spin-1/2 particle can be expressed as a two-element column matrix (or spinor):

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-,$$  \hfill [4.139]

with

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$  \hfill [4.140]

representing spin up, and

$$\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$  \hfill [4.141]

for spin down.

Meanwhile, the spin operators become $2 \times 2$ matrices, which we can work out by noting their effect on $\chi_+$ and $\chi_-$. Equation 4.135 says

$$S^2\chi_+ = \frac{3}{4}\hbar^2\chi_+ \quad \text{and} \quad S^2\chi_- = \frac{3}{4}\hbar^2\chi_-.$$  \hfill [4.142]

If we write $S^2$ as a matrix with (as yet) undetermined elements,

$$S^2 = \begin{pmatrix} c & d \\ e & f \end{pmatrix},$$

then the first equation says

$$\begin{pmatrix} c & d \\ e & f \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

or

$$\begin{pmatrix} c \\ e \end{pmatrix} = \begin{pmatrix} \frac{3}{4}\hbar^2 \\ 0 \end{pmatrix},$$

so $c = (3/4)\hbar^2$ and $e = 0$. The second equation says

$$\begin{pmatrix} c & d \\ e & f \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{3}{4}\hbar^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

or

$$\begin{pmatrix} d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{3}{4}\hbar^2 \end{pmatrix},$$

so $d = 0$ and $f = (3/4)\hbar^2$. Conclusion:

$$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  \hfill [4.143]
Similarly,
\[ S_x \chi_+ = \frac{h}{2} \chi_+ , \quad S_x \chi_- = -\frac{h}{2} \chi_- , \]
from which it follows that
\[ S_z = \frac{h}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \]
[4.145]

Meanwhile, Equation 4.136 says
\[ S_+ \chi_- = h \chi_+ , \quad S_- \chi_+ = h \chi_- , \quad S_+ \chi_+ = S_- \chi_- = 0 , \]
so
\[ S_+ = h \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad S_- = h \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \]
[4.146]

Now \( S_\pm = S_x \pm i S_y \), so \( S_x = (1/2)(S_+ + S_-) \) and \( S_y = (1/2i)(S_+ - S_-) \), and hence
\[ S_x = \frac{h}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad S_y = \frac{h}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \]
[4.147]

Since \( S_x, S_y, \) and \( S_z \) all carry a factor of \( h/2 \), it is tidier to write \( \mathbf{S} = (h/2) \mathbf{\sigma} \), where
\[ \sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \]
[4.148]

These are the famous Pauli spin matrices. Notice that \( S_x, S_y, S_z, \) and \( S^2 \) are all hermitian (as they should be, since they represent observables). On the other hand, \( S_+ \) and \( S_- \) are not hermitian—evidently they are not observable.

The eigenspinors of \( S_z \) are (of course):
\[ \chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \text{(eigenvalue } + \frac{h}{2} \text{)} ; \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad \text{(eigenvalue } - \frac{h}{2} \text{)} . \]
[4.149]

If you measure \( S_z \) on a particle in the general state \( \chi \) (Equation 4.139), you could get \( +h/2 \), with probability \( |a|^2 \), or \( -h/2 \), with probability \( |b|^2 \). Since these are the only possibilities,
\[ |a|^2 + |b|^2 = 1 \]
[4.150]

(i.e., the spinor must be normalized)\(^{29}\)

\(^{29}\)People often say that \( |a|^2 \) is the "probability that the particle is in the spin-up state," but this is sloppy language; what they mean is that if you measured \( S_z \), \( |a|^2 \) is the probability you'd get \( h/2 \). See footnote 16 in Chapter 3.
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But what if, instead, you chose to measure $S_z$? What are the possible results, and what are their respective probabilities? According to the generalized statistical interpretation, we need to know the eigenvalues and eigenspinors of $\hat{S}_z$. The characteristic equation is

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 = \left(\frac{\hbar}{2}\right)^2 \Rightarrow \lambda = \pm \frac{\hbar}{2}.$$

Not surprisingly, the possible values for $S_z$ are the same as those for $S_x$. The eigenspinors are obtained in the usual way:

$$\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

so $\beta = \pm \alpha$. Evidently the (normalized) eigenspinors of $\hat{S}_x$ are

$$\chi^{(+)}_\pm = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \text{(eigenvalue } + \frac{\hbar}{2} \text{)}; \quad \chi^{(-)}_\pm = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \text{(eigenvalue } - \frac{\hbar}{2} \text{)} \quad [4.151]$$

As the eigenvectors of a hermitian matrix, they span the space; the generic spinor $\chi$ (Equation 4.139) can be expressed as a linear combination of them:

$$\chi = \left(\frac{a + b}{\sqrt{2}}\right) \chi^{(+)} + \left(\frac{a - b}{\sqrt{2}}\right) \chi^{(-)} \quad [4.152]$$

If you measure $S_z$, the probability of getting $+\hbar/2$ is $(1/2)|a + b|^2$, and the probability of getting $-\hbar/2$ is $(1/2)|a - b|^2$. (You should check for yourself that these probabilities add up to 1.)

---

Example 4.2  Suppose a spin-1/2 particle is in the state

$$\chi = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 + i \\ 2 \end{pmatrix}.$$

What are the probabilities of getting $+h/2$ and $-h/2$, if you measure $S$ and $S_x$?

Solution: Here $a = (1 + i)/\sqrt{6}$ and $b = 2/\sqrt{6}$, so for $S_z$ the probability of getting $+h/2$ is $|(1 + i)/\sqrt{6}|^2 = 1/3$, and the probability of getting $-h/2$ is $|2/\sqrt{6}|^2 = 2/3$. For $S_x$ the probability of getting $+h/2$ is $(1/2)(3 + i)/\sqrt{6}|^2 = 5/6$, and
the probability of getting \(-\hbar/2\) is \((1/2)((-1 + i)/\sqrt{6})^2 = 1/6\). Incidentally, the expectation value of \(S_z\) is

\[
\frac{5}{6} \left( \frac{\hbar}{2} \right) + \frac{1}{6} \left( -\frac{\hbar}{2} \right) = \frac{\hbar}{3},
\]

which we could also have obtained more directly:

\[
\langle S_z \rangle = \chi_x^* S_z \chi = \left( \frac{1 - i}{\sqrt{6}} \right) \left( \frac{2}{h/2} \right) \left( \begin{array}{cc} 0 & h/2 \\ h/2 & 0 \end{array} \right) \left( \begin{array}{c} (1 + i)/\sqrt{6} \\ 2/\sqrt{6} \end{array} \right) = \frac{\hbar}{3}.
\]

I'd like now to walk you through an imaginary measurement scenario involving spin 1/2, because it serves to illustrate in very concrete terms some of the abstract ideas we discussed back in Chapter 1. Let's say we start out with a particle in the state \(\chi_+\). If someone asks, "What is the z-component of that particle's spin angular momentum?", we could answer unambiguously: \(+\hbar/2\). For a measurement of \(S_z\) is certain to return that value. But if our interrogator asks instead, "What is the x-component of that particle's spin angular momentum?" we are obliged to equivocate: If you measure \(S_x\), the chances are fifty-fifty of getting either \(\hbar/2\) or \(-\hbar/2\). If the questioner is a classical physicist, or a "realist" (in the sense of Section 1.2), he will regard this as an inadequate—not to say impertinent—response: "Are you telling me that you don't know the true state of that particle?" On the contrary; I know precisely what the state of the particle is: \(\chi_+\). "Well, then, how come you can't tell me what the x-component of its spin is?" Because it simply does not have a particular x-component of spin. Indeed, it cannot, for if both \(S_x\) and \(S_z\) were well-defined, the uncertainty principle would be violated.

At this point our challenger grabs the test-tube and measures the x-component of its spin; let's say he gets the value \(+\hbar/2\). "Aha!" (he shouts in triumph), "You lied! This particle has a perfectly well-defined value of \(S_z\): \(+\hbar/2\)." Well, sure—it does now, but that doesn't prove it had that value, prior to your measurement. "You have obviously been reduced to splitting hairs. And anyway, what happened to your uncertainty principle? I now know both \(S_x\) and \(S_z\)." I'm sorry, but you do not: In the course of your measurement, you altered the particle's state; it is now in the state \(\chi_+^{(x)}\), and whereas you know the value of \(S_x\), you no longer know the value of \(S_z\). "But I was extremely careful not to disturb the particle when I measured \(S_x\)." Very well, if you don't believe me, check it out: Measure \(S_z\), and see what you get. (Of course, he may get \(+\hbar/2\), which will be embarrassing to my case—but if we repeat this whole scenario over and over, half the time he will get \(-\hbar/2\).)

To the layman, the philosopher, or the classical physicist, a statement of the form "this particle doesn't have a well-defined position" (or momentum, or x-component of spin angular momentum, or whatever) sounds vague, incompetent, or (worst of all) profound. It is none of these. But its precise meaning is, I think,
almost impossible to convey to anyone who has not studied quantum mechanics in some depth. If you find your own comprehension slipping, from time to time (if you don’t, you probably haven’t understood the problem), come back to the spin-1/2 system: It is the simplest and cleanest context for thinking through the conceptual paradoxes of quantum mechanics.

Problem 4.26

(a) Check that the spin matrices (Equations 4.145 and 4.147) obey the fundamental commutation relations for angular momentum, Equation 4.134.

(b) Show that the Pauli spin matrices (Equation 4.148) satisfy the product rule

$$\sigma_j \sigma_k = \delta_{jk} + i \sum_l \epsilon_{jkl} \sigma_l,$$  \[4.153\]

where the indices stand for $x$, $y$, or $z$, and $\epsilon_{jkl}$ is the Levi-Civita symbol: +1 if $jkl = 123$, 231, or 312; −1 if $jkl = 132$, 213, or 321; 0 otherwise.

*Problem 4.27* An electron is in the spin state

$$\chi = A \begin{pmatrix} 3/4 \end{pmatrix}.$$  

(a) Determine the normalization constant $A$.

(b) Find the expectation values of $S_x$, $S_y$, and $S_z$.

(c) Find the “uncertainties” $\sigma_{S_x}$, $\sigma_{S_y}$, and $\sigma_{S_z}$. (*Note:* These sigmas are standard deviations, not Pauli matrices!)

(d) Confirm that your results are consistent with all three uncertainty principles (Equation 4.100 and its cyclic permutations—only with $S$ in place of $L$, of course).

*Problem 4.28* For the most general normalized spinor $\chi$ (Equation 4.139), compute $\langle S_x \rangle$, $\langle S_y \rangle$, $\langle S_z \rangle$, $\langle S_x^2 \rangle$, $\langle S_y^2 \rangle$, and $\langle S_z^2 \rangle$. Check that $\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \langle S^2 \rangle$.

*Problem 4.29*

(a) Find the eigenvalues and eigenspinors of $S_y$. 

(b) If you measured $S_y$ on a particle in the general state $\chi$ (Equation 4.139), what values might you get, and what is the probability of each? Check that the probabilities add up to 1. Note: $a$ and $b$ need not be real!

(c) If you measured $S_y^2$, what values might you get, and with what probabilities?

**Problem 4.30** Construct the matrix $S_r$ representing the component of spin angular momentum along an arbitrary direction $\hat{r}$. Use spherical coordinates, for which

$$\hat{r} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}.$$  

[4.154]

Find the eigenvalues and (normalized) eigenspinors of $S_r$. *Answer:*

$$\chi_+^{(r)} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix}; \quad \chi_-^{(r)} = \begin{pmatrix} e^{-i\phi} \sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix}.$$  

[4.155]

*Note: You're always free to multiply by an arbitrary phase factor—say, $e^{i\phi}$—so your answer may not look exactly the same as mine.*

**Problem 4.31** Construct the spin matrices ($S_x$, $S_y$, and $S_z$) for a particle of spin 1. *Hint: How many eigenstates of $S_z$ are there? Determine the action of $S_z$, $S_+$, and $S_-$ on each of these states. Follow the procedure used in the text for spin 1/2.*

**4.4.2 Electron in a Magnetic Field**

A spinning charged particle constitutes a magnetic dipole. Its **magnetic dipole moment**, $\mu$, is proportional to its spin angular momentum, $S$:

$$\mu = \gamma S;$$  

[4.156]

the proportionality constant, $\gamma$, is called the **gyromagnetic ratio**.\(^3\) When a magnetic dipole is placed in a magnetic field $B$, it experiences a torque, $\mu \times B$, which

\(^3\) See, for example, D. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice Hall, Upper Saddle River, NJ, 1999), page 252. Classically, the gyromagnetic ratio of an object whose charge and mass are identically distributed is $q/2m$, where $q$ is the charge and $m$ is the mass. For reasons that are fully explained only in relativistic quantum theory, the gyromagnetic ratio of the electron is (almost) exactly twice the classical value: $\gamma = -e/m$. 
tends to line it up parallel to the field (just like a compass needle). The energy associated with this torque is\footnote{Griffiths (footnote 30), page 281.}

\[ H = -\mu \cdot B, \]  \hspace{1cm} \text{[4.157]} 

so the Hamiltonian of a spinning charged particle, at rest\footnote{If the particle is allowed to move, there will also be kinetic energy to consider; moreover, it will be subject to the Lorentz force \((\mathbf{q} \times \mathbf{B})\), which is not derivable from a potential energy function, and hence does not fit the Schrödinger equation as we have formulated it so far. I’ll show you later on how to handle this (Problem 4.59), but for the moment let’s just assume that the particle is free to rotate, but otherwise stationary.} in a magnetic field \( \mathbf{B} \), is

\[ H = -\gamma B \cdot \mathbf{S}. \]  \hspace{1cm} \text{[4.158]}

---

**Example 4.3 Larmor precession:** Imagine a particle of spin \( 1/2 \) at rest in a uniform magnetic field, which points in the \( z \)-direction:

\[ \mathbf{B} = B_0 \hat{k}. \]  \hspace{1cm} \text{[4.159]}

The Hamiltonian (Equation 4.158), in matrix form, is

\[ \mathbf{H} = -\gamma B_0 \mathbf{S}_z = -\frac{\gamma B_0 \hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  \hspace{1cm} \text{[4.160]}

The eigenstates of \( \mathbf{H} \) are the same as those of \( \mathbf{S}_z \):

\[ \begin{cases} x_+, \text{ with energy } E_+ = -\frac{\gamma B_0 \hbar}{2}, \\ x_-, \text{ with energy } E_- = +\frac{\gamma B_0 \hbar}{2}. \end{cases} \]  \hspace{1cm} \text{[4.161]}

Evidently the energy is lowest when the dipole moment is parallel to the field—just as it would be classically.

Since the Hamiltonian is time-independent, the general solution to the time-dependent Schrödinger equation,

\[ i\hbar \frac{\partial \chi}{\partial t} = \mathbf{H} \chi, \]  \hspace{1cm} \text{[4.162]}

can be expressed in terms of the stationary states:

\[ \chi(t) = a x_+ e^{-i E_+ t/\hbar} + b x_- e^{-i E_- t/\hbar} = \begin{pmatrix} a e^{i\gamma B_0 t/\hbar} \\ b e^{-i\gamma B_0 t/\hbar} \end{pmatrix}. \]
The constants $a$ and $b$ are determined by the initial conditions:

$$\chi(0) = \begin{pmatrix} a \\ b \end{pmatrix}.$$  

(of course, $|a|^2 + |b|^2 = 1$). With no essential loss of generality\(^{33}\) I’ll write $a = \cos(\alpha/2)$ and $b = \sin(\alpha/2)$, where $\alpha$ is a fixed angle whose physical significance will appear in a moment. Thus

$$\chi(t) = \begin{pmatrix} \cos(\alpha/2)e^{iy伯/2} \\ \sin(\alpha/2)e^{-iy伯/2} \end{pmatrix}.$$  \[4.163\]

To get a feel for what is happening here, let’s calculate the expectation value of $\mathbf{S}$, as a function of time:

$$\langle S_x \rangle = \chi(t)\dagger \mathbf{S}_x \chi(t) = \begin{pmatrix} \cos(\alpha/2)e^{-iy伯/2} & \sin(\alpha/2)e^{iy伯/2} \end{pmatrix} \times \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\alpha/2)e^{iy伯/2} \\ \sin(\alpha/2)e^{-iy伯/2} \end{pmatrix}$$

$$= \frac{\hbar}{2} \sin \alpha \cos(伯t).$$  \[4.164\]

Similarly,

$$\langle S_y \rangle = \chi(t)\dagger \mathbf{S}_y \chi(t) = -\frac{\hbar}{2} \sin \alpha \sin(伯t),$$  \[4.165\]

and

$$\langle S_z \rangle = \chi(t)\dagger \mathbf{S}_z \chi(t) = \frac{\hbar}{2} \cos \alpha.$$  \[4.166\]

Evidently $(\mathbf{S})$ is tilted at a constant angle $\alpha$ to the $z$-axis, and precesses about the field at the Larmor frequency

$$\omega = y伯_0,$$  \[4.167\]

just as it would classically\(^ {34}\) (see Figure 4.10). No surprise here—Ehrenfest’s theorem (in the form derived in Problem 4.20) guarantees that $(\mathbf{S})$ evolves according to the classical laws. But it’s nice to see how this works out in a specific context.

\(^{33}\)This does assume that $a$ and $b$ are real; you can work out the general case if you like, but all it does is add a constant to $i$.

\(^{34}\)See, for instance, The Feynman Lectures on Physics (Addison-Wesley, Reading, 1964), Volume II, Section 34-3. Of course, in the classical case it is the angular momentum vector itself, not just its expectation value, that precesses around the magnetic field.
Example 4.4  The Stern-Gerlach experiment: In an inhomogeneous magnetic field, there is not only a torque, but also a force, on a magnetic dipole:\(^{35}\)

\[
F = \nabla (\mu \cdot B). \tag{4.168}
\]

This force can be used to separate out particles with a particular spin orientation, as follows. Imagine a beam of relatively heavy neutral atoms,\(^{36}\) traveling in the \(y\) direction, which passes through a region of inhomogeneous magnetic field (Figure 4.11)—say,

\[
B(x, y, z) = -\alpha x \hat{i} + (B_0 + \alpha z) \hat{k}. \tag{4.169}
\]

where \(B_0\) is a strong uniform field and the constant \(\alpha\) describes a small deviation from homogeneity. (Actually, what we’d like is just the \(z\) component, but unfortunately that’s impossible—it would violate the electromagnetic law \(\nabla \cdot B = 0\); like it or not, an \(x\) component comes along for the ride.) The force on these atoms is

\[
F = \gamma \alpha (-S_x \hat{i} + S_z \hat{k}).
\]

\(^{35}\)Griffiths (footnote 30), page 258. Note that \(F\) is the negative gradient of the energy (Equation 4.157).

\(^{36}\)We make them neutral so as to avoid the large-scale deflection that would otherwise result from the Lorentz force, and heavy so we can construct localized wave packets and treat the motion in terms of classical particle trajectories. In practice, the Stern-Gerlach experiment doesn’t work, for example, with a beam of free electrons.
But because of the Larmor precession about \( B_0 \), \( S_x \) oscillates rapidly, and averages to zero; the net force is in the \( z \) direction:

\[
F_z = \gamma \alpha S_z, \tag{4.170}
\]

and the beam is deflected up or down, in proportion to the \( z \) component of the spin angular momentum. Classically we'd expect a smear (because \( S_z \) would not be quantized), but in fact the beam splits into \( 2s + 1 \) separate streams, beautifully demonstrating the quantization of angular momentum. (If you use silver atoms, for example, all the inner electrons are paired, in such a way that their spin and orbital angular momenta cancel. The net spin is simply that of the outermost—unpaired—electron, so in this case \( s = 1/2 \), and the beam splits in two.)

Now, that argument was purely classical, up to the very final step; “force” has no place in a proper quantum calculation, and you might therefore prefer the following approach to the same problem.\(^{37}\) We examine the process from the perspective of a reference frame that moves along with the beam. In this frame the Hamiltonian starts out zero, turns on for a time \( T \) (as the particle passes through the magnet), and then turns off again:

\[
H(t) = \begin{cases} 
0, & \text{for } t < 0, \\
-\gamma (B_0 + \alpha z) S_z, & \text{for } 0 \leq t \leq T, \\
0, & \text{for } t > T.
\end{cases} \tag{4.171}
\]

(I ignore the pesky \( x \) component of \( B \), which—for reasons indicated above—is irrelevant to the problem.) Suppose the atom has spin \( 1/2 \), and starts out in the state

\[
\chi(t) = a \chi_+ + b \chi_-, \quad \text{for } t \leq 0.
\]

\(^{37}\)This argument follows L. Ballentine (footnote 26) Section 9.1.
While the Hamiltonian acts, $\chi(t)$ evolves in the usual way:

$$
\chi(t) = a\chi_+ e^{-iE_+ t/\hbar} + b\chi_- e^{-iE_- t/\hbar}, \quad \text{for } 0 \leq t \leq T,
$$

where (from Equation 4.158)

$$
E_\pm = \mp \gamma (B_0 + \alpha z) \frac{\hbar}{2},
$$

[4.172]

and hence it emerges in the state

$$
\chi(t) = \left( ae^{i\gamma T B_0/2} \chi_+ \right) e^{i(\alpha \gamma T/2)z} + \left( be^{-i\gamma T B_0/2} \chi_- \right) e^{-(i\alpha \gamma T/2)z},
$$

[4.173]

(for $t \geq T$). The two terms now carry momentum in the $z$ direction (see Equation 3.32); the spin-up component has momentum

$$
p_z = \frac{\alpha \gamma T \hbar}{2},
$$

[4.174]

and it moves in the plus-$z$ direction; the spin-down component has the opposite momentum, and it moves in the minus-$z$ direction. Thus the beam splits in two, as before. (Note that Equation 4.174 is consistent with the earlier result (Equation 4.170), for in this case $S_z = \hbar/2$, and $p_z = F_z T$.)

The Stern-Gerlach experiment has played an important role in the philosophy of quantum mechanics, where it serves both as the prototype for the preparation of a quantum state and as an illuminating model for a certain kind of quantum measurement. We tend casually to assume that the initial state of a system is known (the Schrödinger equation tells us how it subsequently evolves)—but it is natural to wonder how you get a system into a particular state in the first place. Well, if you want to prepare a beam of atoms in a given spin configuration, you pass an unpolarized beam through a Stern-Gerlach magnet, and select the outgoing stream you are interested in (closing off the others with suitable baffles and shutters). Conversely, if you want to measure the $z$ component of an atom’s spin, you send it through a Stern-Gerlach apparatus, and record which bin it lands in. I do not claim that this is always the most practical way to do the job, but it is conceptually very clean, and hence a useful context in which to explore the problems of state preparation and measurement.

---

**Problem 4.32** In Example 4.3:

(a) If you measured the component of spin angular momentum along the $x$ direction, at time $t$, what is the probability that you would get $+\hbar/2$?
(b) Same question, but for the $y$ component.

(c) Same, for the $z$ component.

**Problem 4.33** An electron is at rest in an oscillating magnetic field

$$B = B_0 \cos(\omega t) \hat{k},$$

where $B_0$ and $\omega$ are constants.

(a) Construct the Hamiltonian matrix for this system.

(b) The electron starts out (at $t = 0$) in the spin-up state with respect to the $x$-axis (that is: $\chi(0) = \chi^{(s)}$). Determine $\chi(t)$ at any subsequent time. **Beware:** This is a time-dependent Hamiltonian, so you cannot get $\chi(t)$ in the usual way from stationary states. Fortunately, in this case you can solve the time-dependent Schrödinger equation (Equation 4.162) directly.

(c) Find the probability of getting $-\hbar/2$, if you measure $S_x$. **Answer:**

$$\sin^2 \left( \frac{\gamma B_0}{2\omega} \sin(\omega t) \right).$$

(d) What is the minimum field ($B_0$) required to force a complete flip in $S_x$?

4.4.3 Addition of Angular Momenta

Suppose now that we have two spin-1/2 particles—for example, the electron and the proton in the ground state\textsuperscript{38} of hydrogen. Each can have spin up or spin down, so there are four possibilities in all:\textsuperscript{39}

$$\uparrow \uparrow, \uparrow \downarrow, \downarrow \uparrow, \downarrow \downarrow. \quad [4.175]$$

where the first arrow refers to the electron and the second to the proton. **Question:** What is the total angular momentum of the atom? Let

$$S = S^{(1)} + S^{(2)}. \quad [4.176]$$

---

\textsuperscript{38} I put them in the ground state so there won't be any orbital angular momentum to worry about.

\textsuperscript{39} More precisely, each particle is in a linear combination of spin up and spin down, and the composite system is in a linear combination of the four states listed.
Each of these four composite states is an eigenstate of $S_z$—the $z$ components simply add:

$$S_z X_1 X_2 = (S_z^{(1)} + S_z^{(2)}) X_1 X_2 = (S_z^{(1)} X_1) X_2 + X_1 (S_z^{(2)} X_2)$$

$$= (\hbar m_1 X_1) X_2 + X_1 (\hbar m_2 X_2) = \hbar (m_1 + m_2) X_1 X_2,$$

(note that $S_z^{(1)}$ acts only on $X_1$, and $S_z^{(2)}$ acts only on $X_2$; this notation may not be elegant, but it does the job). So $m$ (the quantum number for the composite system) is just $m_1 + m_2$:

\[
\begin{align*}
\uparrow\uparrow & : m = 1; \\
\uparrow\downarrow & : m = 0; \\
\downarrow\uparrow & : m = 0; \\
\downarrow\downarrow & : m = -1.
\end{align*}
\]

At first glance, this doesn’t look right: $m$ is supposed to advance in integer steps, from $-s$ to $+s$, so it appears that $s = 1$—but there is an “extra” state with $m = 0$. One way to untangle this problem is to apply the lowering operator, $S_- = S_-^{(1)} + S_-^{(2)}$ to the state $\uparrow\uparrow$, using Equation 4.146:

$$S_- (\uparrow\uparrow) = (S_-^{(1)} \uparrow) \uparrow + (S_-^{(2)} \uparrow)$$

$$= (h \downarrow) \uparrow + (\hbar \downarrow) = h (\downarrow \uparrow + \uparrow \downarrow).$$

Evidently the three states with $s = 1$ are (in the notation $|s m\rangle$):

\[
\begin{align*}
|1 1\rangle & = \uparrow\uparrow \\
|1 0\rangle & = \frac{1}{\sqrt{2}} (\uparrow\downarrow + \downarrow\uparrow) \\
|1 -1\rangle & = \downarrow\downarrow
\end{align*}
\]

[4.177]

(As a check, try applying the lowering operator to $|1 0\rangle$; what should you get? See Problem 4.34(a).) This is called the triplet combination, for the obvious reason. Meanwhile, the orthogonal state with $m = 0$ carries $s = 0$:

\[
\begin{align*}
|0 0\rangle & = \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \\
\end{align*}
\]

[4.178]

(If you apply the raising or lowering operator to this state, you’ll get zero. See Problem 4.34(b).)
I claim, then, that the combination of two spin-1/2 particles can carry a total spin of 1 or 0, depending on whether they occupy the triplet or the singlet configuration. To confirm this, I need to prove that the triplet states are eigenvectors of $S^2$ with eigenvalue $2h^2$, and the singlet is an eigenvector of $S^2$ with eigenvalue 0. Now,

$$S^2 = (S^{(1)} + S^{(2)}) \cdot (S^{(1)} + S^{(2)}) = (S^{(1)})^2 + (S^{(2)})^2 + 2S^{(1)} \cdot S^{(2)}.$$  \hspace{1cm} [4.179]$$

Using Equations 4.145 and 4.147, we have

$$S^{(1)} \cdot S^{(2)} (\uparrow \downarrow) = (S^{(1)}_x \uparrow)(S^{(2)}_x \downarrow) + (S^{(1)}_y \uparrow)(S^{(2)}_y \downarrow) + (S^{(1)}_z \uparrow)(S^{(2)}_z \downarrow)$$

$$= \left( \frac{\hbar}{2} \downarrow \right) \left( \frac{\hbar}{2} \uparrow \right) + \left( \frac{i\hbar}{2} \downarrow \right) \left( -\frac{i\hbar}{2} \uparrow \right) + \left( \frac{\hbar}{2} \uparrow \right) \left( -\frac{\hbar}{2} \downarrow \right)$$

$$= \frac{\hbar^2}{4} (2 \downarrow \uparrow - \uparrow \downarrow).$$

Similarly,

$$S^{(1)} \cdot S^{(2)} (\downarrow \uparrow) = \frac{\hbar^2}{4} (2 \uparrow \downarrow - \downarrow \uparrow).$$

It follows that

$$S^{(1)} \cdot S^{(2)} |10\rangle = \frac{\hbar^2}{4} \frac{1}{\sqrt{2}} (2 \downarrow \uparrow - \uparrow \downarrow + 2 \uparrow \downarrow - \downarrow \uparrow) = \frac{\hbar^2}{4} |10\rangle,$$  \hspace{1cm} [4.180]$$

and

$$S^{(1)} \cdot S^{(2)} |00\rangle = \frac{\hbar^2}{4} \frac{1}{\sqrt{2}} (2 \downarrow \uparrow - \uparrow \downarrow - 2 \uparrow \downarrow + \downarrow \uparrow) = -\frac{3\hbar^2}{4} |00\rangle.$$  \hspace{1cm} [4.181]$$

Returning to Equation 4.179 (and using Equation 4.142), we conclude that

$$S^2 |10\rangle = \left( \frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + \frac{\hbar^2}{4} \right) |10\rangle = 2\hbar^2 |10\rangle,$$  \hspace{1cm} [4.182]$$

so $|10\rangle$ is indeed an eigenstate of $S^2$ with eigenvalue $2\hbar^2$; and

$$S^2 |00\rangle = \left( \frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} - \frac{3\hbar^2}{4} \right) |00\rangle = 0,$$  \hspace{1cm} [4.183]$$

so $|00\rangle$ is an eigenstate of $S^2$ with eigenvalue 0. (I will leave it for you to confirm that $|1\uparrow\rangle$ and $|1\downarrow\rangle$ are eigenstates of $S^2$, with the appropriate eigenvalue—see Problem 4.34(c).)
Section 4.4: Spin

What we have just done (combining spin $1/2$ with spin $1/2$ to get spin $1$ and spin $0$) is the simplest example of a larger problem: If you combine spin $s_1$ with spin $s_2$, what total spins $s$ can you get? The answer is that you get every spin from $(s_1 + s_2)$ down to $(s_1 - s_2)$—or $(s_2 - s_1)$, if $s_2 > s_1$—in integer steps:

$$s = (s_1 + s_2), (s_1 + s_2 - 1), (s_1 + s_2 - 2), \ldots, |s_1 - s_2|. \quad [4.184]$$

(Roughly speaking, the highest total spin occurs when the individual spins are aligned parallel to one another, and the lowest occurs when they are antiparallel.) For example, if you package together a particle of spin $3/2$ with a particle of spin $2$, you could get a total spin of $7/2, 5/2, 3/2, 1/2$, depending on the configuration. Another example: If a hydrogen atom is in the state $\psi_{nlm}$, the net angular momentum of the electron (spin plus orbital) is $l + 1/2$ or $l - 1/2$; if you now throw in spin of the proton, the atom’s total angular momentum quantum number is $l + 1$, $l$, or $l - 1$ (and $l$ can be achieved in two distinct ways, depending on whether the electron alone is in the $l + 1/2$ configuration or the $l - 1/2$ configuration).

The combined state $|s m\rangle$ with total spin $s$ and $z$-component $m$ will be some linear combination of the composite states $|s_1 m_1 \rangle |s_2 m_2 \rangle$:

$$|s m\rangle = \sum_{m_1+m_2=m} C_{m_1 m_2 m}^{s_1 s_2} |s_1 m_1 \rangle |s_2 m_2 \rangle \quad [4.185]$$

(because the $z$ components add, the only composite states that contribute are those for which $m_1 + m_2 = m$). Equations 4.177 and 4.178 are special cases of this general form, with $s_1 = s_2 = 1/2$ (I used the informal notation $\uparrow = |1/2\rangle$, $\downarrow = |1/2\rangle$). The constants $C_{m_1 m_2 m}^{s_1 s_2}$ are called Clebsch-Gordan coefficients. A few of the simplest cases are listed in Table 4.8. For example, the shaded column of the $2 \times 1$ table tells us that

$$|3 0\rangle = \frac{1}{\sqrt{2}}|2 1\rangle|1 - 1\rangle + \sqrt{\frac{3}{2}}|2 0\rangle|1 0\rangle + \frac{1}{\sqrt{3}}|2 - 1\rangle|1 1\rangle.$$

In particular, if two particles (of spin $2$ and spin $1$) are at rest in a box, and the total spin is $3$, and its $z$ component is $0$, then a measurement of $S_z$ could return the value $h$ (with probability $1/5$), or $0$ (with probability $3/5$), or $-h$ (with probability $1/5$).

---

40 For spins, for simplicity, but either one (or both) could just as well be orbital angular momentum (for which, however, we would use the letter $l$).

41 For a proof you must look in a more advanced text; see, for instance, Claude Cohen-Tannoudji, Bernard Diu, and Franck Laloe, Quantum Mechanics, (Wiley, New York, 1977), Vol. 2, Chapter X.

TABLE 4.8: Clebsch-Gordan coefficients. (A square root sign is understood for every entry; the minus sign, if present, goes outside the radical.)

1/5). Notice that the probabilities add up to 1 (the sum of the squares of any column on the Clebsch-Gordan table is 1).

These tables also work the other way around:

$$|s_1 m_1 \rangle |s_2 m_2 \rangle = \sum_s C_{m_1 m_2 m}^{s_1 s_2} |s m \rangle.$$  \[4.186\]

For example, the shaded row in the 3/2 × 1 table tells us that

$$|\frac{3}{2} \frac{1}{2} \rangle |1 0 \rangle = \sqrt{\frac{3}{5}} |\frac{3}{2} \frac{1}{2} \rangle + \sqrt{\frac{1}{15}} |\frac{3}{2} \frac{1}{2} \rangle - \sqrt{\frac{1}{15}} |\frac{1}{2} \frac{1}{2} \rangle.$$

If you put particles of spin 3/2 and spin 1 in the box, and you know that the first has $m_1 = 1/2$ and the second has $m_2 = 0$ (so $m$ is necessarily 1/2), and you measure the total spin, $s$, you could get 5/2 (with probability 3/5), or 3/2 (with probability 1/15), or 1/2 (with probability 1/3). Again, the sum of the probabilities is 1 (the sum of the squares of each row on the Clebsch-Gordan table is 1).

If you think this is starting to sound like mystical numerology, I don’t blame you. We will not be using the Clebsch-Gordan tables much in the rest of the book, but I wanted you to know where they fit into the scheme of things, in case you encounter them later on. In a mathematical sense this is all applied group theory—what we are talking about is the decomposition of the direct product of
two irreducible representations of the rotation group into a direct sum of irreducible representations (you can quote that, to impress your friends).

*Problem 4.34*

(a) Apply $S_-$ to $|10\rangle$ (Equation 4.177), and confirm that you get $\sqrt{2}\hbar|1-1\rangle$.
(b) Apply $S_\pm$ to $|00\rangle$ (Equation 4.178), and confirm that you get zero.
(c) Show that $|11\rangle$ and $|1-1\rangle$ (Equation 4.177) are eigenstates of $S^2$, with the appropriate eigenvalue.

**Problem 4.35** Quarks carry spin 1/2. Three quarks bind together to make a baryon (such as the proton or neutron); two quarks (or more precisely a quark and an antiquark) bind together to make a meson (such as the pion or the kaon). Assume the quarks are in the ground state (so the orbital angular momentum is zero).

(a) What spins are possible for baryons?
(b) What spins are possible for mesons?

**Problem 4.36**

(a) A particle of spin 1 and a particle of spin 2 are at rest in a configuration such that the total spin is 3, and its $z$ component is $\hbar$. If you measured the $z$ component of the angular momentum of the spin-2 particle, what values might you get, and what is the probability of each one?
(b) An electron with spin down is in the state $\psi_{5\,10}$ of the hydrogen atom. If you could measure the total angular momentum squared of the electron alone (not including the proton spin), what values might you get, and what is the probability of each?

**Problem 4.37** Determine the commutator of $S^2$ with $S_z^{(1)}$ (where $S \equiv S^{(1)} + S^{(2)}$). Generalize your result to show that

$$[S^2, S^{(1)}] = 2i\hbar(S^{(1)} \times S^{(2)}).$$

[4.187]

*Comment:* Because $S_z^{(1)}$ does not commute with $S^2$, we cannot hope to find states that are simultaneous eigenvectors of both. In order to form eigenstates of $S^2$ we