Solutions to TISE

The solutions to the Time-Independent Schrödinger’s Equation have several interesting properties that we need to discuss before we actually find solutions to different potential functions.

1) The solutions $\psi(x)$ represent states of the particle that give expectation values that do not change with time. We can show this by looking at the probability density $\Psi^*\Psi$:

$$\Psi^*(x, t)\Psi(x, t) = (\psi(x)\varphi(t))^*(\psi(x)\varphi(t)) = \left(\psi^*(x)e^{\frac{+iE}{\hbar}t}\right)\left(\psi(x)e^{-\frac{-iE}{\hbar}t}\right) = \psi^*(x)\psi(x)$$

with no time dependence. And, since all dynamical operators can be expressed as functions of the position and momentum operators,

$$\int \Psi^*(x, t)\hat{O}(\tilde{x}, \tilde{p})\Psi(x, t)dx = \int (\psi(x)\varphi(t))^*\hat{O}(\tilde{x}, \tilde{p})(\psi(x)\varphi(t))dx$$

$$= \int \left(\psi^*(x)e^{\frac{+iE}{\hbar}t}\right)\hat{O}(x, \frac{\hbar}{i}\frac{d}{dx})\left(\psi(x)e^{-\frac{-iE}{\hbar}t}\right)dx$$

$$= \int \psi^*(x)\hat{O}(x, \frac{\hbar}{i}\frac{d}{dx})\psi(x)dx$$

Therefore, all expectation values do not change with time. Remember: the wave function $\Psi(x,t)$ does change with time, but the expectation values are constant, which, from our discussion of $<p>$ earlier means that if $<x>$ is constant, $<p> = 0$. These $\psi$ are then called stationary states. Now, you should be disturbed. How can these represent real particles if nothing ever happens? We will get to that soon.

2) The solutions $\psi(x)$ represent states of the particle that have definite total energy $E$.

This is subtly but importantly different than expanding the previous statement to include the expectation value of the energy. Let’s begin there though:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

but the TISE tells us that:

$$\hat{H}\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

so that the expectation value is given by:
\[
\langle H \rangle = \int \psi^*(x) \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) \psi(x) dx
\]
\[
= \int \psi^*(x) E \psi(x) dx
\]
\[
= E \int \psi^*(x) \psi(x) dx
\]
\[
= E
\]

As we would expect, but also

\[
\langle H^2 \rangle = \int \psi^*(x) \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right)^2 \psi(x) dx
\]
\[
= \int \psi^*(x) \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right) E \psi(x) dx
\]
\[
= \int \psi^*(x) E^2 \psi(x) dx
\]
\[
= E^2
\]

so that when we look at the variance:

\[
\sigma_H^2 = \langle H^2 \rangle - \langle H \rangle^2 = 0
\]

but, for the variance to be zero, every measurement of the energy must be the same, namely, \( E \).

3) The general solution for a particular potential is a linear combination of the stationary state solutions. The TISE has, in general an infinite number of stationary state solutions, \( \psi_1 \), \( \psi_2 \), etc., each having a different separation constant \( E \). Now, one of the solutions to the time-dependent Schrödinger’s equation is of the form:

\[
\Psi_1^1 (x, t) = \psi_1 (x) \varphi_1 (t) = \psi_1 (x) e^{-\frac{iE_1}{\hbar} t}
\]

and it can be shown that the general solution to the time-dependent Schrödinger’s equation is a linear combination of the of these solutions:

\[
\Psi(x, t) = \sum_{n=1}^{\infty} c_n \Psi_n (x, t) = \sum_{n=1}^{\infty} c_n \psi_n (x) \varphi_n (t) = \sum_{n=1}^{\infty} c_n \psi_n (x) e^{-\frac{iE_n}{\hbar} t}
\]
where the $c_n$ are any complex constants. It is important to remember that the separable solutions are stationary states, giving constant measurables, but the general solution is not, since the energies are different, and the exponents do not cancel in $\Psi^* \Psi$.

There are several very important and subtle points here, so we should spend a little time discussing this.

**Properties of $\psi(x)$**

The solutions to the TISE must behave in certain ways. We list those properties here and will then immediately show some exceptions to the rules…

1) **$\psi(x)$ must be finite.** This is derived from the normalizability of the wave function and from the requirement that dynamic quantities that we extract from $\psi(x)$ also be finite.

   **Exception:** In some cases $\psi(x)$ may go to infinity at a point as long as it approaches infinity slowly enough such that the integral of $\psi(x)^* \psi(x)$ over an interval containing that point remains finite.

2) **$\psi(x)$ must be single valued.** If $\psi(x)$ is not single valued, then the expectation values one derives from it are undetermined, which is not desirable.

3) **$\psi(x)$ must be continuous.** If $\psi(x)$ is not continuous, then $d\psi(x)/dx$ is not finite, and the momentum would go to infinity at that point.

4) **$d\psi(x)/dx$ must be finite.** As above, else the momentum would not be finite.

5) **$d\psi(x)/dx$ must be single valued.** Again, as above, then the momentum would not be single-valued.

6) **$d\psi(x)/dx$ must be continuous.** If the derivative is not continuous, then the second derivative is infinite, and from the TISE, it would require that either the energy or the potential be infinite.

   **Exception:** If the potential is infinite, then the first derivative must not be continuous. We will see this shortly.