CHAPTER 11

SCATTERING

11.1 INTRODUCTION

11.1.1 Classical Scattering Theory

Imagine a particle incident on some scattering center (say, a proton fired at a heavy nucleus). It comes in with energy E and **impact parameter** b, and it emerges at some **scattering angle** θ—see Figure 11.1. (I'll assume for simplicity that the target is azimuthally symmetrical, so the trajectory remains in one plane, and that the target is very heavy, so the reçoil is negligible.) The essential problem of classical scattering theory is this: *Given the impact parameter, calculate the scattering angle.* Ordinarily, of course, the smaller the impact parameter, the greater the scattering angle.

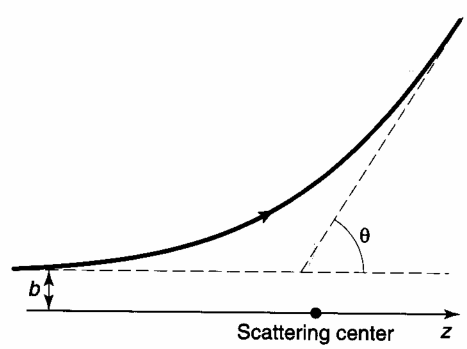

FIGURE 11.1: The classical scattering problem, showing the impact parameter b and the scattering angle θ.

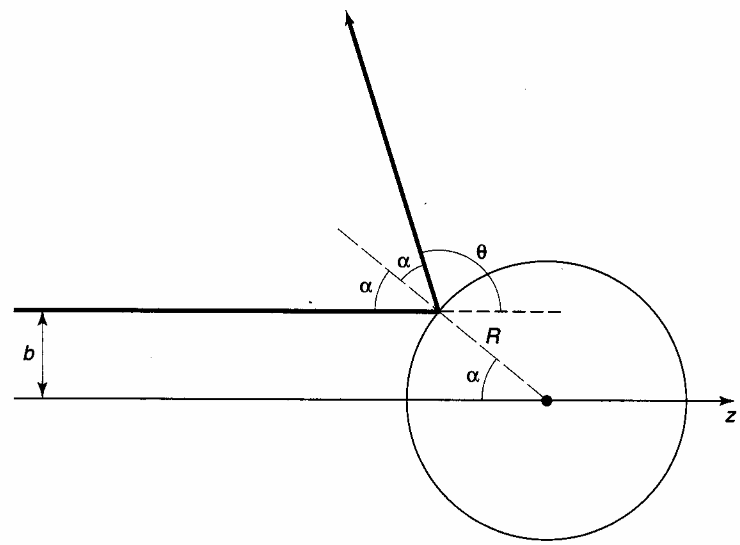

FIGURE 11.2: **Elastic hard-sphere scattering.**

Example 11.1 **Hard-sphere scattering.** Suppose the target is a billiard ball, of radius R, and the incident particle is a BB, which bounces off elastically (Figure 11.2). In terms of the angle α, the impact parameter is $b = R\sin\alpha$, and the scattering angle is $\theta = \pi - 2\alpha$, so

$$b = R\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = R\cos\left(\frac{\theta}{2}\right). \qquad [11.1]$$

Evidently

$$\theta = \begin{cases} 2\cos^{-1}(b/R), & \text{if } b \leq R, \\ 0, & \text{if } b \geq R. \end{cases} \qquad [11.2]$$

More generally, particles incident within an infinitesimal patch of cross-sectional area $d\sigma$ will scatter into a corresponding infinitesimal solid angle $d\Omega$ (Figure 11.3). The larger $d\sigma$ is, the bigger $d\Omega$ will be; the proportionality factor, $D(\theta) \equiv d\sigma/d\Omega$, is called the **differential (scattering) cross-section:**[1]

[1]This is terrible language: D isn't a *differential*, and it isn't a cross-section. To my ear, the words "differential cross-section" would attach more naturally to $d\sigma$. But I'm afraid we're stuck with this terminology. I should also warn you that the notation $D(\theta)$ is nonstandard: Most people just call it $d\sigma/d\Omega$—which makes Equation 11.3 look like a tautology. I think it will be less confusing if we give the differential cross-section its own symbol.

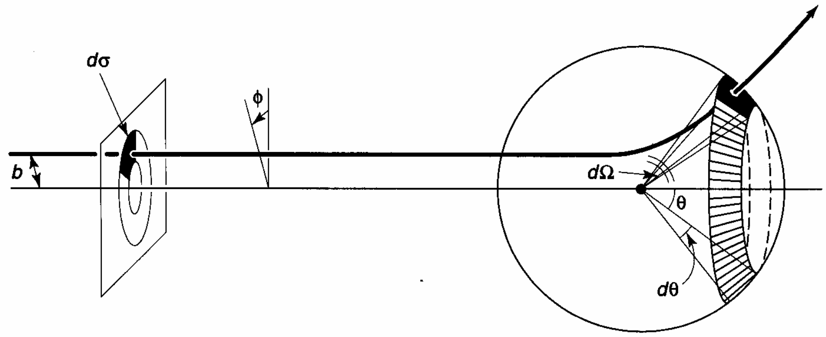

FIGURE 11.3: Particles incident in the area $d\sigma$ scatter into the solid angle $d\Omega$.

$$\boxed{d\sigma = D(\theta)\, d\Omega.} \quad [11.3]$$

In terms of the impact parameter and the azimuthal angle ϕ, $d\sigma = b\, db\, d\phi$ and $d\Omega = \sin\theta\, d\theta\, d\phi$, so

$$D(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|. \quad [11.4]$$

(Since θ is typically a *decreasing* function of b, the derivative is actually negative—hence the absolute value sign.)

Example 11.2 Hard-sphere scattering (continued). In the case of hard-sphere scattering (Example 11.1)

$$\frac{db}{d\theta} = -\frac{1}{2}R\sin\left(\frac{\theta}{2}\right), \quad [11.5]$$

so

$$D(\theta) = \frac{R\cos(\theta/2)}{\sin\theta}\left(\frac{R\sin(\theta/2)}{2}\right) = \frac{R^2}{4}. \quad [11.6]$$

This example is unusual, in that the differential cross-section is independent of θ.

The **total cross-section** is the *integral* of $D(\theta)$, over all solid angles:

$$\boxed{\sigma \equiv \int D(\theta)\, d\Omega;} \quad [11.7]$$

roughly speaking, it is the total area of incident beam that is scattered by the target. For example, in the case of hard-sphere scattering,

$$\sigma = (R^2/4) \int d\Omega = \pi R^2, \qquad [11.8]$$

which is just what we would expect: It's the cross-sectional area of the sphere; BB's incident within this area will hit the target, and those farther out will miss it completely. But the virtue of the formalism developed here is that it applies just as well to "soft" targets (such as the Coulomb field of a nucleus) that are *not* simply "hit-or-miss."

Finally, suppose we have a *beam* of incident particles, with uniform intensity (or **luminosity**, as particle physicists call it)

$$\mathcal{L} \equiv \text{number of incident particles per unit area, per unit time.} \qquad [11.9]$$

The number of particles entering area $d\sigma$ (and hence scattering into solid angle $d\Omega$), per unit time, is $dN = \mathcal{L}\, d\sigma = \mathcal{L}\, D(\theta)\, d\Omega$, so

$$D(\theta) = \frac{1}{\mathcal{L}} \frac{dN}{d\Omega}. \qquad [11.10]$$

This is often taken as the *definition* of the differential cross-section, because it makes reference only to quantities easily measured in the laboratory: If the detector accepts particles scattering into a solid angle $d\Omega$, we simply count the *number* recorded, per unit time, divide by $d\Omega$, and normalize to the luminosity of the incident beam.

$***$**Problem 11.1 Rutherford scattering.** An incident particle of charge q_1 and kinetic energy E scatters off a heavy stationary particle of charge q_2.

(a) Derive the formula relating the impact parameter to the scattering angle.[2] *Answer:* $b = (q_1 q_2 / 8\pi\epsilon_0 E)\cot(\theta/2)$.

(b) Determine the differential scattering cross-section. *Answer:*

$$D(\theta) = \left[\frac{q_1 q_2}{16\pi\epsilon_0 E \sin^2(\theta/2)}\right]^2. \qquad [11.11]$$

(c) Show that the total cross-section for Rutherford scattering is *infinite*. We say that the $1/r$ potential has "infinite range"; you can't escape from a Coulomb force.

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[2]This isn't easy, and you might want to refer to a book on classical mechanics, such as Jerry B. Marion and Stephen T. Thornton, *Classical Dynamics of Particles and Systems*, 4th ed., Saunders, Fort Worth, TX (1995), Section 9.10.

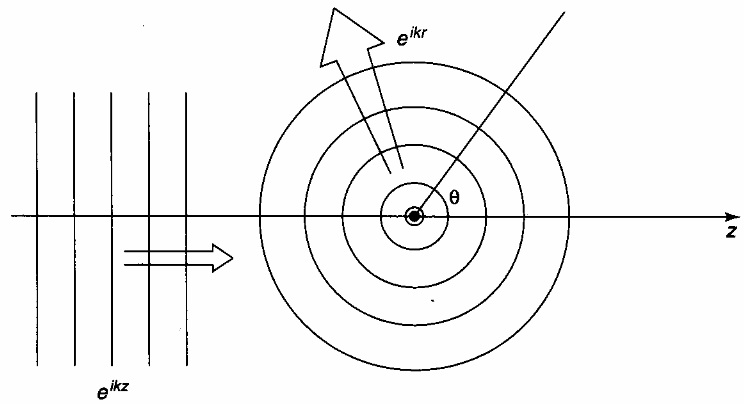

FIGURE 11.4: Scattering of waves; incoming plane wave generates outgoing spherical wave.

11.1.2 Quantum Scattering Theory

In the quantum theory of scattering, we imagine an incident *plane* wave, $\psi(z) = Ae^{ikz}$, traveling in the z direction, which encounters a scattering potential, producing an outgoing *spherical* wave (Figure 11.4).[3] That is, we look for solutions to the Schrödinger equation of the general form

$$\boxed{\psi(r,\theta) \approx A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\}, \quad \text{for large } r.} \qquad [11.12]$$

(The spherical wave carries a factor of $1/r$, because this portion of $|\psi|^2$ must go like $1/r^2$ to conserve probability.) The **wave number** k is related to the energy of the incident particles in the usual way:

$$k \equiv \frac{\sqrt{2mE}}{\hbar}. \qquad [11.13]$$

As before, I shall assume the target is azimuthally symmetrical; in the more general case the amplitude f of the outgoing spherical wave could depend on ϕ as well as θ.

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[3]For the moment, there's not much *quantum* mechanics in this; what we're really talking about is the scattering of *waves*, as opposed to classical *particles*, and you could even think of Figure 11.4 as a picture of water waves encountering a rock, or (better, since we're interested in three-dimensional scattering) sound waves bouncing off a basketball. In that case we'd write the wave function in the *real* form

$$A\left[\cos(kz) + f(\theta)\cos(kr+\delta)/r\right],$$

and $f(\theta)$ would represent the amplitude of the scattered sound wave in the direction θ.

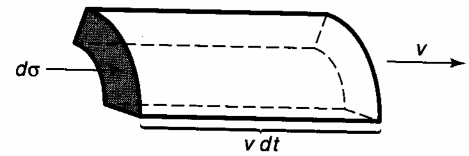

FIGURE 11.5: The volume dV of incident beam that passes through area $d\sigma$ in time dt.

The whole problem is to determine the **scattering amplitude** $f(\theta)$; it tells you the *probability of scattering in a given direction* θ, and hence is related to the differential cross-section. Indeed, the probability that the incident particle, traveling at speed v, passes through the infinitesimal area $d\sigma$, in time dt, is (see Figure 11.5)

$$dP = |\psi_{\text{incident}}|^2\, dV = |A|^2 (v\, dt)\, d\sigma.$$

But this is equal to the probability that the particle scatters into the corresponding solid angle $d\Omega$:

$$dP = |\psi_{\text{scattered}}|^2\, dV = \frac{|A|^2|f|^2}{r^2}(v\, dt) r^2\, d\Omega,$$

from which it follows that $d\sigma = |f|^2\, d\Omega$, and hence

$$\boxed{D(\theta) = \frac{d\sigma}{d\Omega} = |f(\theta)|^2.} \qquad [11.14]$$

Evidently the differential cross-section (which is the quantity of interest to the experimentalist) is equal to the absolute square of the scattering amplitude (which is obtained by solving the Schrödinger equation). In the following sections we will study two techniques for calculating the scattering amplitude: **partial wave analysis** and the **Born approximation**.

Problem 11.2 Construct the analogs to Equation 11.12 for one-dimensional and two-dimensional scattering.

11.2 PARTIAL WAVE ANALYSIS

11.2.1 Formalism

As we found in Chapter 4, the Schrödinger equation for a spherically symmetrical potential $V(r)$ admits the separable solutions

$$\psi(r, \theta, \phi) = R(r) Y_l^m(\theta, \phi), \qquad [11.15]$$

where Y_l^m is a spherical harmonic (Equation 4.32), and $u(r) = rR(r)$ satisfies the radial equation (Equation 4.37):

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu. \qquad [11.16]$$

At *very* large r the potential goes to zero, and the centrifugal contribution is negligible, so

$$\frac{d^2u}{dr^2} \approx -k^2u.$$

The general solution is

$$u(r) = Ce^{ikr} + De^{-ikr};$$

the first term represents an *outgoing* spherical wave, and the second an *incoming* one—for the scattered wave we evidently want $D = 0$. At very large r, then,

$$R(r) \sim \frac{e^{ikr}}{r},$$

as we already deduced (on physical grounds) in the previous section (Equation 11.12).

That's for *very* large r (more precisely, for $kr \gg 1$; in optics it would be called the **radiation zone**). As in one-dimensional scattering theory, we assume that the potential is "localized," in the sense that exterior to some finite scattering region it is essentially zero (Figure 11.6). In the intermediate region (where V can be ignored but the centrifugal term cannot),[4] the radial equation becomes

$$\frac{d^2u}{dr^2} - \frac{l(l+1)}{r^2}u = -k^2u, \qquad [11.17]$$

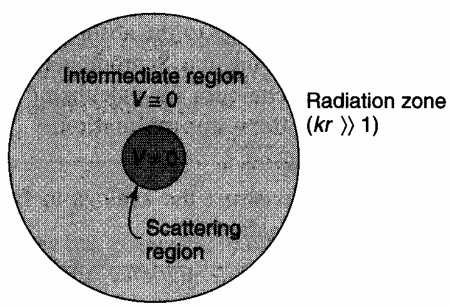

FIGURE 11.6: Scattering from a localized potential: the scattering region (darker shading), the intermediate region (lighter shading), and the radiation zone (where $kr \gg 1$).

[4]What follows does not apply to the Coulomb potential, since $1/r$ goes to zero more slowly than $1/r^2$, as $r \to \infty$, and the centrifugal term does *not* dominate in this region. In this sense the Coulomb potential is not localized, and partial wave analysis is inapplicable.

TABLE 11.1: Spherical Hankel functions, $h_l^{(1)}(x)$ and $h_l^{(2)}(x)$.

$h_0^{(1)} = -i\frac{e^{ix}}{x}$	$h_0^{(2)} = i\frac{e^{-ix}}{x}$
$h_1^{(1)} = \left(-\frac{i}{x^2} - \frac{1}{x}\right)e^{ix}$	$h_1^{(2)} = \left(\frac{i}{x^2} - \frac{1}{x}\right)e^{-ix}$
$h_2^{(1)} = \left(-\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right)e^{ix}$	$h_2^{(2)} = \left(\frac{3i}{x^3} - \frac{3}{x^2} + \frac{i}{x}\right)e^{-ix}$
$h_l^{(1)} \rightarrow \frac{1}{x}(-i)^{l+1}e^{ix}$ $h_l^{(2)} \rightarrow \frac{1}{x}(i)^{l+1}e^{-ix}$	for $x \gg 1$

and the general solution (Equation 4.45) is a linear combination of spherical Bessel functions:

$$u(r) = Arj_l(kr) + Brn_l(kr). \quad [11.18]$$

However, neither j_l (which is somewhat like a sine function) nor n_l (which is a sort of generalized cosine function) represents an outgoing (or an incoming) wave. What we need are the linear combinations analogous to e^{ikr} and e^{-ikr}; these are known as **spherical Hankel functions**:

$$h_l^{(1)}(x) \equiv j_l(x) + in_l(x); \quad h_l^{(2)}(x) \equiv j_l(x) - in_l(x). \quad [11.19]$$

The first few spherical Hankel functions are listed in Table 11.1. At large r, $h_l^{(1)}(kr)$ (the "Hankel function of the first kind") goes like e^{ikr}/r, whereas $h_l^{(2)}(kr)$ (the "Hankel function of the second kind") goes like e^{-ikr}/r; for outgoing waves, then, we need *spherical Hankel functions of the first kind:*

$$R(r) \sim h_l^{(1)}(kr). \quad [11.20]$$

Thus the exact wave function, outside the scattering region (where $V(r) = 0$), is

$$\psi(r, \theta, \phi) = A\left\{e^{ikz} + \sum_{l,m} C_{l,m}\, h_l^{(1)}(kr)\, Y_l^m(\theta, \phi)\right\}. \quad [11.21]$$

The first term is the incident plane wave, and the sum (with expansion coefficients $C_{l,m}$) represents the scattered wave. But since we are assuming the potential is spherically symmetric, the wave function cannot depend on ϕ.[5] So only terms with

[5]There's nothing wrong with θ dependence, of course, because the incoming plane wave defines a z direction, breaking the spherical symmetry. But the *azimuthal* symmetry remains; the incident plane wave has no ϕ dependence, and there is nothing in the scattering process that could introduce any ϕ dependence in the outgoing wave.

$m = 0$ survive (remember, $Y_l^m \sim e^{im\phi}$). Now (from Equations 4.27 and 4.32)

$$Y_l^0(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta), \qquad [11.22]$$

where P_l is the lth Legendre polynomial. It is customary to redefine the expansion coefficients, letting $C_{l,0} \equiv i^{l+1}k\sqrt{4\pi(2l+1)}\, a_l$:

$$\boxed{\psi(r,\theta) = A\left\{e^{ikz} + k\sum_{l=0}^{\infty} i^{l+1}(2l+1)\, a_l\, h_l^{(1)}(kr)\, P_l(\cos\theta)\right\}.} \qquad [11.23]$$

You'll see in a moment why this peculiar notation is convenient; a_l is called the lth **partial wave amplitude**.

Now, for *very large* r the Hankel function goes like $(-i)^{l+1}e^{ikr}/kr$ (Table 11.1), so

$$\psi(r,\theta) \approx A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\}, \qquad [11.24]$$

where

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)\, a_l\, P_l(\cos\theta). \qquad [11.25]$$

This confirms more rigorously the general structure postulated in Equation 11.12, and tells us how to compute the scattering amplitude, $f(\theta)$, in terms of the partial wave amplitudes (a_l). The differential cross-section is

$$D(\theta) = |f(\theta)|^2 = \sum_l\sum_{l'}(2l+1)(2l'+1)\, a_l^*\, a_{l'}\, P_l(\cos\theta)\, P_{l'}(\cos\theta), \qquad [11.26]$$

and the total cross-section is

$$\sigma = 4\pi\sum_{l=0}^{\infty}(2l+1)\,|a_l|^2. \qquad [11.27]$$

(I used the orthogonality of the Legendre polynomials, Equation 4.34, to do the angular integration.)

11.2.2 Strategy

All that remains is to determine the partial wave amplitudes, a_l, for the potential in question. This is accomplished by solving the Schrödinger equation in the *interior* region (where $V(r)$ is distinctly *non*-zero), and matching this to the exterior solution (Equation 11.23), using the appropriate boundary conditions. The only problem is

In particular, the total cross-section is

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\left|\frac{j_l(ka)}{h_l^{(1)}(ka)}\right|^2. \qquad [11.34]$$

That's the *exact* answer, but it's not terribly illuminating, so let's consider the limiting case of *low-energy scattering*: $ka \ll 1$. (Since $k = 2\pi/\lambda$, this amounts to saying that the wavelength is much greater than the radius of the sphere.) Referring to Table 4.4, we note that $n_l(z)$ is much larger than $j_l(z)$, for small z, so

$$\frac{j_l(z)}{h_l^{(1)}(z)} = \frac{j_l(z)}{j_l(z) + i n_l(z)} \approx -i\frac{j_l(z)}{n_l(z)}$$

$$\approx -i\frac{2^l l! z^l/(2l+1)!}{-(2l)! z^{-l-1}/2^l l!} = \frac{i}{2l+1}\left[\frac{2^l l!}{(2l)!}\right]^2 z^{2l+1}, \qquad [11.35]$$

and hence

$$\sigma \approx \frac{4\pi}{k^2}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[\frac{2^l l!}{(2l)!}\right]^4 (ka)^{4l+2}.$$

But we're assuming $ka \ll 1$, so the higher powers are negligible—in the low energy approximation the scattering is dominated by the $l = 0$ term. (This means that the differential cross-section is independent of θ, just as it was in the classical case.) Evidently

$$\sigma \approx 4\pi a^2, \qquad [11.36]$$

for low energy hard-sphere scattering. Surprisingly, the scattering cross-section is *four times* the geometrical cross-section—in fact, σ is the *total surface area of the sphere*. This "larger effective size" is characteristic of long-wavelength scattering (it would be true in optics, as well); in a sense, these waves "feel" their way around the whole sphere, whereas classical *particles* only see the head-on cross-section.

Problem 11.3 Prove Equation 11.33, starting with Equation 11.32. *Hint:* Exploit the orthogonality of the Legendre polynomials to show that the coefficients with different values of l must separately vanish.

∗∗Problem 11.4 Consider the case of low-energy scattering from a spherical delta-function shell:

$$V(r) = \alpha\delta(r - a),$$

where α and a are constants. Calculate the scattering amplitude, $f(\theta)$, the differential cross-section, $D(\theta)$, and the total cross-section, σ. Assume $ka \ll 1$, so that only the $l = 0$ term contributes significantly. (To simplify matters, throw out all $l \neq 0$ terms right from the start.) The main problem, of course, is to determine a_0. Express your answer in terms of the dimensionless quantity $\beta \equiv 2ma\alpha/\hbar^2$. *Answer:* $\sigma = 4\pi a^2\beta^2/(1+\beta)^2$.

11.3 PHASE SHIFTS

Consider first the problem of *one*-dimensional scattering from a localized potential $V(x)$ on the half-line $x < 0$ (Figure 11.7). I'll put a "brick wall" at $x = 0$, so a wave incident from the left,

$$\psi_i(x) = Ae^{ikx} \quad (x < -a) \tag{11.37}$$

is entirely reflected

$$\psi_r(x) = Be^{-ikx} \quad (x < -a). \tag{11.38}$$

Whatever happens in the interaction region $(-a < x < 0)$, the amplitude of the reflected wave has *got* to be the same as that of the incident wave, by conservation of probability. But it need not have the same *phase*. If there were no potential at all (just the wall at $x = 0$), then $B = -A$, since the total wave function (incident plus reflected) must vanish at the origin:

$$\psi_0(x) = A\left(e^{ikx} - e^{-ikx}\right) \quad (V(x) = 0). \tag{11.39}$$

If the potential is *not* zero, the wave function (for $x < -a$) takes the form

$$\psi(x) = A\left(e^{ikx} - e^{i(2\delta - kx)}\right) \quad (V(x) \neq 0). \tag{11.40}$$

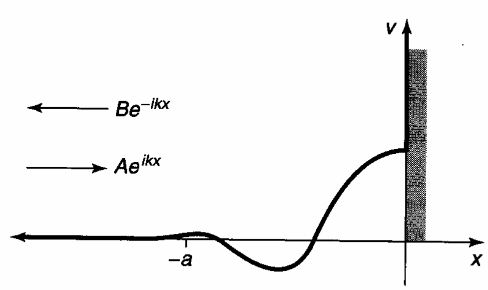

FIGURE 11.7: One-dimensional scattering from a localized potential bounded on the right by an infinite wall.

The whole theory of scattering reduces to the problem of calculating the **phase shift**[7] δ (as a function of k, and hence of the energy $E = \hbar^2k^2/2m$), for a specified potential. We do this, of course, by solving the Schrödinger equation in the scattering region ($-a < x < 0$), and imposing appropriate boundary conditions (see Problem 11.5). The virtue of working with the phase shift (as opposed to the complex amplitude B) is that it illuminates the physics (because of conservation of probability, *all* the potential can do is shift the phase of the reflected wave) and simplifies the mathematics (trading a complex quantity—two real numbers—for a single real quantity).

Now let's return to the three-dimensional case. The incident plane wave (Ae^{ikz}) carries no angular momentum in the z direction (Rayleigh's formula contains no terms with $m \neq 0$), but it includes all values of the *total* angular momentum ($l = 0, 1, 2, \ldots$). Because angular momentum is conserved (by a spherically symmetric potential), each **partial wave** (labelled by a particular l) scatters independently, with (again) no change in amplitude[8]—only in phase. If there is no potential at all, then $\psi_0 = Ae^{ikz}$, and the lth partial wave is (Equation 11.28)

$$\psi_0^{(l)} = Ai^l(2l+1)\, j_l(kr)\, P_l(\cos\theta) \quad (V(r) = 0). \qquad [11.41]$$

But (from Equation 11.19 and Table 11.1)

$$j_l(x) = \frac{1}{2}\left[h^{(1)}(x) + h_l^{(2)}(x)\right] \approx \frac{1}{2x}\left[(-i)^{l+1}e^{ix} + i^{l+1}e^{-ix}\right] \quad (x \gg 1). \qquad [11.42]$$

So for large r

$$\psi_0^{(l)} \approx A\frac{(2l+1)}{2ikr}\left[e^{ikr} - (-1)^l e^{-ikr}\right] P_l(\cos\theta) \quad (V(r) = 0). \qquad [11.43]$$

The second term in square brackets represents an incoming spherical wave; it is unchanged when we introduce the scattering potential. The first term is the outgoing wave; it picks up a phase shift δ_l:

$$\psi^{(l)} \approx A\frac{(2l+1)}{2ikr}\left[e^{i(kr+2\delta_l)} - (-1)^l e^{-ikr}\right] P_l(\cos\theta) \quad (V(r) \neq 0). \qquad [11.44]$$

Think of it as a converging spherical wave (due exclusively to the $h_l^{(2)}$ component in e^{ikz}), which is phase shifted $2\delta_l$ (see footnote 7) and emerges as an outgoing spherical wave (the $h_l^{(1)}$ part of e^{ikz} as well as the scattered wave itself).

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[7] The 2 in front of δ is conventional. We think of the incident wave as being phase shifted once on the way in, and again on the way out; by δ we mean the "one way" phase shift, and the *total* is therefore 2δ.

[8] One reason this subject can be so confusing is that practically everything is called an "amplitude:" $f(\theta)$ is the "scattering amplitude," a_l is the "partial wave amplitude," but the first is a function of θ, and both are complex numbers. I'm *now* talking about "amplitude" in the original sense: the (*real*, of course) height of a sinusoidal wave.

In Section 11.2.1 the whole theory was expressed in terms of the partial wave amplitudes a_l; now we have formulated it in terms of the phase shifts δ_l. There must be a connection between the two. Indeed, comparing the asymptotic (large r) form of Equation 11.23

$$\psi^{(l)} \approx A\left\{\frac{(2l+1)}{2ikr}\left[e^{ikr} - (-1)^l e^{-ikr}\right] + \frac{(2l+1)}{r} a_l e^{ikr}\right\} P_l(\cos\theta) \qquad [11.45]$$

with the generic expression in terms of δ_l (Equation 11.44), we find[9]

$$a_l = \frac{1}{2ik}\left(e^{2i\delta_l} - 1\right) = \frac{1}{k} e^{i\delta_l} \sin(\delta_l). \qquad [11.46]$$

It follows in particular (Equation 11.25) that

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin(\delta_l)\, P_l(\cos\theta) \qquad [11.47]$$

and (Equation 11.27)

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_l). \qquad [11.48]$$

Again, the advantage of working with phase shifts (as opposed to partial wave amplitudes) is that they are easier to interpret physically, and simpler mathematically—the phase shift formalism exploits conservation of angular momentum to reduce a complex quantity a_l (two real numbers) to a single real one δ_l.

Problem 11.5 A particle of mass m and energy E is incident from the left on the potential

$$V(x) = \begin{cases} 0, & (x < -a), \\ -V_0, & (-a \le x \le 0), \\ \infty, & (x > 0). \end{cases}$$

(a) If the incoming wave is Ae^{ikx} (where $k = \sqrt{2mE}/\hbar$), find the reflected wave. *Answer:*

$$Ae^{-2ika}\left[\frac{k - ik'\cot(k'a)}{k + ik'\cot(k'a)}\right]e^{-ikx}, \quad \text{where } k' = \sqrt{2m(E+V_0)}/\hbar.$$

[9] Although I used the asymptotic form of the wave function to draw the connection between a_l and δ_l, there is nothing approximate about the result (Equation 11.46). Both of them are *constants* (independent of r), and δ_l *means* the phase shift in the asymptotic region (where the Hankel functions have settled down to $e^{\pm ikr}/kr$).