Classical and Quantum Trajectories

Let’s take a closer look at the behavior of a quantum system given a potential, and the energy of the particle. Shown below is some part of a potential, \( V(x) \), and, on the same scale, the particle’s energy \( E \). The regions of space are divided into two separate domains: 1) where \( E > V(x) \) and 2) where \( E < V(x) \). The boundary between these two regions is where \( E = V(x) \).

![Potential and Energy Graph]

Now, classically, the particle is forbidden to be in region 2, and the boundary where \( E = V(x) \) is the classical turning point. The Schrödinger’s equation:

\[
\psi'' - \frac{\hbar^2}{2m} \psi = -V(x) \psi + E \psi
\]

can be re-written as:

\[
\psi'' - \frac{\hbar^2}{2m} \psi = [E - V(x)] \psi
\]

where we can identify \( E-V(x) \) as being the kinetic energy. Classically, the kinetic energy is restricted to be positive, but there is no such restriction given by the Schrödinger’s equation.

As we have discussed before, the signs of the terms on the left hand side and the right hand side must be the same, determining the sign of the wave function and the curvature of the wavefunction depending on the sign of the kinetic energy. In the picture below, the upper two plots are for region 1 where the kinetic energy is positive. Here, because of the way the wavefunction is curved down when it is positive, and curved up when it is negative, oscillations in the wavefunction are allowed. The larger the difference between the energy and the potential, the greater the curvature, and hence the more oscillations there will be. (Remember how the higher energy eigenfunctions of the infinite square-well potential had more nodes.) In region two, since we must have continuity of the wavefunction, there can be no oscillatory motion, and the wavefunction must just decay.
to zero. Notice the big difference to classical physics here: the wavefunction (and therefore the probability to find the particle there) is NOT zero, but goes to zero.

![Graphs](image)

Given this then, one can draw a wavefunction schematically for a simple potential by just knowing the energy and understanding the correspondence between the kinetic energy and the curvature of the wavefunction, and also by obeying the continuity conditions on the wavefunction. I show an example below.

![Graph](image)

In quantum mechanics, there are three general classifications of trajectories. These are demonstrated below. Remembering that where $E = V(x)$ there is a turning point, then for $E = E_1$, the particle is trapped between the turning points (marked with x’s on the
potential graph), and the trajectory is called bound. The particle just oscillates between the two turning points (although the wavefunction CAN “spill” out into the classically forbidden region).

If $E = E_3$, The particle is free, except if it encounters the potential barrier from the right, and scatters off of it. In this particular case, the particle will completely “bounce” off the potential and continue on moving to the right.

If $E = E_2$, there are two different types of trajectories, bound inside the “well”, and scattering if the particles moves to the left from large $x$. We will also discover that there is some level of “un-boundedness” to the trajectory even when the particle is initially in the well.