Matrix Algebra

The beginnings of matrices and determinants goes back to the second century BC although traces can be seen back to the fourth century BC. However it was not until near the end of the 17th Century that the ideas reappeared and development really got underway.

It is not surprising that the beginnings of matrices and determinants should arise through the study of systems of linear equations. The Babylonians studied problems which lead to simultaneous linear equations and some of these are preserved in clay tablets which survive. For example a tablet dating from around 300 BC contains the following problem:-

There are two fields whose total area is 1800 *square yards. One produces grain at the rate of* $^2/_3$ *of a bushel per square yard while the other produces grain at the rate of* $^1/_2$ *a bushel per square yard. If the total yield is* 1100 *bushels, what is the size of each field.*

The Chinese, between 200 BC and 100 BC, came much closer to matrices than the Babylonians. Indeed it is fair to say that the text Nine Chapters on the Mathematical Art written during the Han Dynasty gives the first known example of matrix methods. First a problem is set up which is similar to the Babylonian example given above:-

There are three types of corn, of which three bundles of the first, two of the second, and one of the third make 39 *measures. Two of the first, three of the second and one of the third make* 34 *measures. And one of the first, two of the second and three of the third make* 26 *measures. How many measures of corn are contained of one bundle of each type?*

Now the author does something quite remarkable. He sets up the coefficients of the system of three linear equations in three unknowns as a table on a 'counting board'.

```
 1    2    3

 2    3    2

 3    1    1

26   34   39
```

Our late 20th Century methods would have us write the linear equations as the rows of the matrix rather than the columns but of course the method is identical. Most remarkably the author, writing in 200 BC, instructs the reader to multiply the middle column by 3 and subtract the right column *as many times as possible*, the same is then done subtracting the right column *as many times as possible* from 3 times the first column. This gives

```
 0    0    3

 4    5    2

 8    1    1

39   24   39
```

Next the left most column is multiplied by 5 and then the middle column is subtracted *as many times as possible*. This gives

```
 0    0    3

 0    5    2

36    1    1

99   24   39
```

from which the solution can be found for the third type of corn, then for the second, then the first by back substitution. This method, now known as Gaussian elimination, would not become well known until the early 19th Century.

Cardan, in *Ars Magna* (1545), gives a rule for solving a system of two linear equations which he calls *regula de modo* and which [7] calls *mother of rules* ! This rule gives what essentially is Cramer's rule for solving a 2 ⁄ 2 system although Cardan does not make the final step. Cardan therefore does not reach the definition of a determinant but, with the advantage of hindsight, we can see that his method does lead to the definition.

Many standard results of elementary matrix theory first appeared long before matrices were the object of mathematical investigation. For example de Witt in *Elements of curves*, published as a part of the commentaries on the 1660 Latin version of Descartes'

Géométrie , showed how a transformation of the axes reduces a given equation for a conic to canonical form. This amounts to diagonalising a symmetric matrix but de Witt never thought in these terms.

The idea of a determinant appeared in Japan and Europe at almost exactly the same time although Seki in Japan certainly published first. In 1683 Seki wrote *Method of solving the dissimulated problems* which contains matrix methods written as tables in exactly the way the Chinese methods described above were constructed. Without having any word which corresponds to 'determinant' Seki still introduced determinants and gave general methods for calculating them based on examples. Using his 'determinants' Seki was able to find determinants of 2×2, 3×3, 4×4 and 5×5 matrices and applied them to solving equations but not systems of linear equations.

Rather remarkably the first appearance of a determinant in Europe appeared in exactly the same year 1683. In that year Leibniz wrote to de l'Hôpital. He explained that the system of equations

```
10 + 11x + 12y = 0

20 + 21x + 22y = 0

30 + 31x + 32y = 0
```

had a solution because

$10.21.32 + 11.22.30 + 12.20.31 = 10.22.31 + 11.20.32 + 12.21.30$

which is exactly the condition that the coefficient matrix has determinant 0. Notice that here Leibniz is not using numerical coefficients but

two characters, the first marking in which equation it occurs, the second marking which letter it belongs to.

Hence 21 denotes what we might write as a_{21}.

Leibniz was convinced that good mathematical notation was the key to progress so he experimented with different notation for coefficient systems. His unpublished manuscripts contain more than 50 different ways of writing coefficient systems which he worked on during a period of 50 years beginning in 1678. Only two publications (1700 and 1710) contain results on coefficient systems and these use the same notation as in his letter to de l'Hôpital mentioned above.

Leibniz used the word 'resultant' for certain combinatorial sums of terms of a determinant. He proved various results on resultants including what is essentially Cramer's rule. He also knew that a determinant could be expanded using any column -

what is now called the [Laplace](#) expansion. As well as studying coefficient systems of equations which led him to determinants, [Leibniz](#) also studied coefficient systems of quadratic forms which led naturally towards matrix theory.

In the 1730's [Maclaurin](#) wrote *Treatise of algebra* although it was not published until 1748, two years after his death. It contains the first published results on determinants proving [Cramer](#)'s rule for 2×2 and 3×3 systems and indicating how the 4×4 case would work. [Cramer](#) gave the general rule for $n \times n$ systems in a paper *Introduction to the analysis of algebraic curves* (1750). It arose out of a desire to find the equation of a plane curve passing through a number of given points. The rule appears in an Appendix to the paper but no proof is given:-

One finds the value of each unknown by forming n fractions of which the common denominator has as many terms as there are permutations of n things.

[Cramer](#) does go on to explain precisely how one calculates these terms as products of certain coefficients in the equations and how one determines the sign. He also says how the n numerators of the fractions can be found by replacing certain coefficients in this calculation by constant terms of the system.

Work on determinants now began to appear regularly. In 1764 [Bezout](#) gave methods of calculating determinants as did [Vandermonde](#) in 1771. In 1772 [Laplace](#) claimed that the methods introduced by [Cramer](#) and [Bezout](#) were impractical and, in a paper where he studied the orbits of the inner planets, he discussed the solution of systems of linear equations without actually calculating it, by using determinants. Rather surprisingly [Laplace](#) used the word 'resultant' for what we now call the determinant: surprising since it is the same word as used by [Leibniz](#) yet [Laplace](#) must have been unaware of [Leibniz](#)'s work. [Laplace](#) gave the expansion of a determinant which is now named after him.

[Lagrange](#), in a paper of 1773, studied identities for 3×3 functional determinants. However this comment is made with hindsight since [Lagrange](#) himself saw no connection between his work and that of [Laplace](#) and [Vandermonde](#). This 1773 paper on mechanics, however, contains what we now think of as the volume interpretation of a determinant for the first time. [Lagrange](#) showed that the tetrahedron formed by O(0,0,0) and the three points $M(x,y,z)$, $M'(x',y',z')$, $M''(x'',y'',z'')$ has volume

$\frac{1}{6}$ $[z(x'y'' - y'x'') + z'(yx'' - xy'') + z''(xy' - yx')]$.

The term 'determinant' was first introduced by [Gauss](#) in *Disquisitiones arithmeticae* (1801) while discussing quadratic forms. He used the term because the determinant determines the properties of the quadratic form. However the concept is not the same as that of our determinant. In the same work [Gauss](#) lays out the coefficients of his quadratic forms in rectangular arrays. He describes matrix multiplication (which he thinks of as composition so he has not yet reached the concept of matrix algebra) and the inverse of a matrix in the particular context of the arrays of coefficients of quadratic forms.

Gaussian elimination, which first appeared in the text [Nine Chapters on the Mathematical Art](#) written in 200 BC, was used by [Gauss](#) in his work which studied the orbit of the asteroid Pallas. Using observations of Pallas taken between 1803 and 1809, [Gauss](#) obtained a system of six linear equations in six unknowns. [Gauss](#) gave a systematic method for solving such equations which is precisely Gaussian elimination on the coefficient matrix.

It was [Cauchy](#) in 1812 who used 'determinant' in its modern sense. [Cauchy](#)'s work is the most complete of the early works on determinants. He reproved the earlier results and gave new results of his own on minors and adjoints. In the 1812 paper the multiplication theorem for determinants is proved for the first time although, at the same meeting of the Institut de France, [Binet](#) also read a paper which contained a proof of the multiplication theorem but it was less satisfactory than that given by [Cauchy](#).

In 1826 [Cauchy](#), in the context of quadratic forms in n variables, used the term 'tableau' for the matrix of coefficients. He found the eigenvalues and gave results on diagonalisation of a matrix in the context of converting a form to the sum of squares. [Cauchy](#) also introduced the idea of similar matrices (but not the term) and showed that if two matrices are similar they have the same characteristic equation. He also, again in the context of quadratic forms, proved that every real symmetric matrix is diagonalisable.

Jacques [Sturm](#) gave a generalisation of the eigenvalue problem in the context of solving systems of ordinary differential equations. In fact the concept of an eigenvalue appeared 80 years earlier, again in work on systems of linear differential equations, by [D'Alembert](#) studying the motion of a string with masses attached to it at various points.

It should be stressed that neither [Cauchy](#) nor Jacques [Sturm](#) realised the generality of the ideas they were introducing and saw them only in the specific contexts in which they were working. [Jacobi](#) from around 1830 and then [Kronecker](#) and [Weierstrass](#) in the 1850's and 1860's also looked at matrix results but again in a special context, this time the notion of a linear transformation. [Jacobi](#) published three treatises on determinants in 1841. These were important in that for the first time the definition of the determinant was made in an algorithmic way and the entries in the determinant were not specified so his results applied equally well to cases were the entries were numbers or to where they were functions. These three papers by [Jacobi](#) made the idea of a determinant widely known.

[Cayley](#), also writing in 1841, published the first English contribution to the theory of determinants. In this paper he used two vertical lines on either side of the array to denote the determinant, a notation which has now become standard.

[Eisenstein](#) in 1844 denoted linear substitutions by a single letter and showed how to add and multiply them like ordinary numbers except for the lack of commutativity. It is fair to say that [Eisenstein](#) was the first to think of linear substitutions as forming an algebra as can be seen in this quote from his 1844 paper:-

An algorithm for calculation can be based on this, it consists of applying the usual rules for the operations of multiplication, division, and exponentiation to symbolic equations between linear systems, correct symbolic equations are always obtained, the sole consideration being that the order of the factors may not be altered.

The first to use the term 'matrix' was Sylvester in 1850. Sylvester defined a matrix to be *an oblong arrangement of terms* and saw it as something which led to various determinants from square arrays contained within it. After leaving America and returning to England in 1851, Sylvester became a lawyer and met Cayley, a fellow lawyer who shared his interest in mathematics. Cayley quickly saw the significance of the matrix concept and by 1853 Cayley had published a note giving, for the first time, the inverse of a matrix.

Cayley in 1858 published *Memoir on the theory of matrices* which is remarkable for containing the first abstract definition of a matrix. He shows that the coefficient arrays studied earlier for quadratic forms and for linear transformations are special cases of his general concept. Cayley gave a matrix algebra defining addition, multiplication, scalar multiplication and inverses. He gave an explicit construction of the inverse of a matrix in terms of the determinant of the matrix. Cayley also proved that, in the case of 2×2 matrices, that a matrix satisfies its own characteristic equation. He stated that he had checked the result for 3×3 matrices, indicating its proof, but says:-

I have not thought it necessary to undertake the labour of a formal proof of the theorem in the general case of a matrix of any degree.

That a matrix satisfies its own characteristic equation is called the Cayley-Hamilton theorem so its reasonable to ask what it has to do with Hamilton. In fact he also proved a special case of the theorem, the 4×4 case, in the course of his investigations into quaternions.

In 1870 the Jordan canonical form appeared in *Treatise on substitutions and algebraic equations* by Jordan. It appears in the context of a canonical form for linear substitutions over the finite field of order a prime.

Frobenius, in 1878, wrote an important work on matrices *On linear substitutions and bilinear forms* although he seemed unaware of Cayley's work. Frobenius in this paper deals with coefficients of forms and does not use the term matrix. However he proved important results on canonical matrices as representatives of equivalence classes of matrices. He cites Kronecker and Weierstrass as having considered special cases of his results in 1874 and 1868 respectively. Frobenius also proved the general result that a matrix satisfies its characteristic equation. This 1878 paper by Frobenius also contains the definition of the rank of a matrix which he used in his work on canonical forms and the definition of orthogonal matrices.

The nullity of a square matrix was defined by Sylvester in 1884. He defined the nullity of A, $n(A)$, to be the largest i such that every minor of A of order $n-i+1$ is zero. Sylvester was

interested in invariants of matrices, that is properties which are not changed by certain transformations. [Sylvester](#) proved that

$$\max\{n(A), n(B)\} \leqslant n(AB) \leqslant n(A) + n(B).$$

In 1896 [Frobenius](#) became aware of [Cayley](#)'s 1858 *Memoir on the theory of matrices* and after this started to use the term matrix. Despite the fact that [Cayley](#) only proved the [Cayley](#)-[Hamilton](#) theorem for 2 ˅ 2 and 3 ˅ 3 matrices, [Frobenius](#) generously attributed the result to [Cayley](#) despite the fact that [Frobenius](#) had been the first to prove the general theorem.

An axiomatic definition of a determinant was used by [Weierstrass](#) in his lectures and, after his death, it was published in 1903 in the note *On determinant theory*. In the same year [Kronecker](#)'s lectures on determinants were also published, again after his death. With these two publications the modern theory of determinants was in place but matrix theory took slightly longer to become a fully accepted theory. An important early text which brought matrices into their proper place within mathematics was *Introduction to higher algebra* by [Bôcher](#) in 1907. [Turnbull](#) and [Aitken](#) wrote influential texts in the 1930's and [Mirsky](#)'s *An introduction to linear algebra* in 1955 saw matrix theory reach its present major role in as one of the most important undergraduate mathematics topic.

[References](#) (13 books/articles)

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An introduction to the theory and application of matrices

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(http://www.morello.co.uk/matrixalgebra.htm)

1 Introduction

This document serves as an introduction to the basics of matrix algebra, and the application of matrices to solving common problems. We produced this guide primarily to support of MatrixCalculator software for Windows, but we hope it will be useful to others too.

You can download a trial version of the program from our website, www.morello.co.uk

What is a matrix?

A matrix is a rectangular array of numbers (called elements), consisting of *m* rows and *n* columns:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & . & . & a_{1,n} \\ a_{2,1} & a_{2,2} & . & . & a_{2,n} \\ . & . & & & . \\ . & . & & & . \\ a_{m,1} & a_{m,2} & . & . & a_{m,n} \end{pmatrix}$$

This is said to be a matrix of *order* m by n. For instance, here is a *2 by 3* matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

If you are from a C/C++ programming background, you will notice that a matrix is very much like a 2 dimensional array, but beware – the indices start from 1 (whereas C array indices start from zero). Also, note that the ordering is row index followed by column index, where a programmer might naturally put the column index first. This is just the way matrix algebra is traditionally written.

Matrices are useful in a solving a number of problems. After we have described the basic algebra of matrices in the sections below, we will go on to give a couple of real life examples.

2 Matrix Algebra

Addition

It is only possible to add 2 matrices if they are of the same order (ie same number of rows and columns). If we add matrices A and B to give a result X, then each element of X is simply the sum of the 2 corresponding elements of A and B,

$$x_{i,j} = a_{i,j} + b_{i,j}$$

for example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 5 & 7 \end{pmatrix}$$

Since matrix addition is performed simply by adding the individual elements, clearly you will get the same result whatever order you add the matrices in. In maths jargon we say that the operation is both commutative and associative:

$$A + B = B + A$$
$$(A + B) + C = A + (B + C)$$

Subtraction

Subtraction of 2 matrices is analogous with addition:

$$x_{i,j} = a_{i,j} - b_{i,j}$$

for example, using the same A and B as before

$$A - B = \begin{pmatrix} 1 & 3 & 3 \\ 3 & 5 & 5 \end{pmatrix}$$

Scalar Multiplication

Scalar multiplication, ie multiplying a matrix by a number (eg F) is simply a matter of multiplying each element of the matrix by the number:

$$x_{i,j} = F \times a_{j,i}$$

For example:

$$3 \times A = 3 \times \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$$

It should be clear from the above that scalar multiplication is commutative, ie

$$3 \times A = A \times 3$$

and also that scalar multiplication is distributive over addition and subtraction, ie

$$3 \times (A + B) = 3 \times A + 3 \times B$$

Matrix Multiplication

We can multiply 2 matrices to give a matrix result:

$$X = A \times B$$

It is only possible to multiply A and B if the number of columns of A is equal to the number of rows of B (A and B are then said to be *conformable*). If A is an *n by m* matrix, and B is an *m by p* matrix, then X will be a *n by p* matrix. The definition of X is:

$$x_{i,j} = \sum_{k=1}^{m} a_{i,k} \times b_{k,j}$$

For example, we can multiply a *2 by 3* matrix and a *3 by 2* matrix, resulting in a 2 by 2 matrix:

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix} \times \begin{pmatrix} 0 & 6 \\ 2 & 8 \\ 4 & 10 \end{pmatrix} = \begin{pmatrix} 1 \times 0 + 3 \times 2 + 5 \times 4 & 1 \times 6 + 3 \times 8 + 5 \times 10 \\ 7 \times 0 + 9 \times 2 + 11 \times 4 & 7 \times 6 + 9 \times 8 + 11 \times 10 \end{pmatrix} = \begin{pmatrix} 26 & 80 \\ 62 & 224 \end{pmatrix}$$

Notice what happens if we change the order of the 2 matrices. This time we are multiplying a *3 by 2* matrix with a *2 by 3* matrix, and the result is a *3 by 3* matrix, quite different from the previous result:

$$\begin{pmatrix} 0 & 6 \\ 2 & 8 \\ 4 & 10 \end{pmatrix} \times \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix} = \begin{pmatrix} 42 & 54 & 66 \\ 58 & 78 & 98 \\ 74 & 102 & 130 \end{pmatrix}$$

This illustrates that matrix multiplication is not commutative. In fact, if you exchange the matrices, the multiplication may become invalid due to conformability. For example, if A

is a *2 by 3* matrix and B is a *3 by 3* matrix, it is possible to for the product AB, but not BA.

Matrix multiplication is, however, associative and distributive. In summary:

$$A \times B \neq B \times A$$
$$(A \times B) \times C = A \times (B \times C)$$
$$(A + B) \times C = A \times C + B \times C$$

Transposition

Transposing a matrix means converting and *m by n* matrix into an *n by m* matrix, by "flipping" the rows and columns.

$$x_{i,j} = a_{j,i}$$

It is denoted by a superscript T, eg:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

As an aside, there is an interesting relationship between transposition and multiplication:

$$(A \times B)^T = B^T \times A^T$$

If you are interested, you can prove this for yourself fairly easily. Hint – look at the definition of matrix multiply, and try swapping the subscripts!

Equality

2 matrices are considered to be *equal* if they are of the same order, and if all their corresponding elements are equal.

3 Special Types of Matrix

Vector

A row vector is a matrix containing a single row, eg

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

A column vector is a matrix containing a single column, eg

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Both of these forms can be used to represent vector quantities which can be manipulated by matrix algebra, see later. To convert between these forms, simply transpose the matrix.

Zero (Null) Matrix

A zero, or null, matrix is one where every element is zero, eg

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Square Matrix

A square matrix is one where the number of rows and columns are equal, eg a *2 by 2* matrix, a *3 by 3* matrix etc.

Diagonal Matrix

A diagonal matrix is a square matrix in which all the elements are zero except for the elements on the leading diagonal, eg:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Unit Matrix

A unit matrix is a square matrix in which all the elements on the leading diagonal are 1, and all the other elements are 0, eg:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The reason that this is called a unit matrix is that if you multiply any matrix by a unit matrix (of the correct size), you will get back the original matrix. A unit matrix is often denoted by I (identity matrix).

Symmetric Matrix

A symmetrix matrix is a square matrix where

$$a_{i,j} = a_{j,i}$$

for all elements. Ie, the matrix is symmetrical about the leading diagonal. For example

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

Skew Symmetric Matrix

A skew symmetrix matrix is a square matrix where

$$a_{i,j} = -a_{j,i}$$

for all elements. Ie, the matrix is anti-symmetrical about the leading diagonal. This of course requires that elements along the diagonal must be zero. For example

$$\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{pmatrix}$$

Orthogonal Matrix

An orthogonal matrix is a square matrix which produces a unit matrix if it is multiplied by its own transpose. Ie:

$$A \times A^T = I$$

4 Inverse Matrices and Determinants

The Inverse of a Matrix

The inverse (or reciprocal) of a square matrix is denoted by the A^{-1}, and is defined by

$$A \times A^{-1} = I$$

For example

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} -0.4 & 0.2 & 0.6 \\ -0.2 & 0.6 & -0.2 \\ 0.8 & -0.4 & -0.2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The 2 matrices on the left are inverses of each other, whose product is the <u>unit</u> matrix. Not all matrices have an inverse, and those which don't are called singular matrices.

Inverting a matrix is a very useful technique, and we will see, but how is it done? Unfortunately it is slightly more complicated than the basic matrix algebra of the previous chapters, so we will need to take a slight detour into the areas of determinants, cofactors and adjoint matrices first.

Determinants

In this section we are simply going to define the determinant and, in later sections, point out some of its properties. A deeper discussion of determinants probably deserves its own paper.

The determinant of a <u>square</u> matrix is a single number calculated by combining all the elements of the matrix. For example, the determinant of a *2 by 2* matrix is

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1} \times a_{2,2} - a_{2,1} \times a_{1,2}$$

For a *3 by 3* matrix the formula is

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1} \times \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{2,1} \times \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} + a_{3,1} \times \begin{vmatrix} a_{1,2} & a_{1,3} \\ a_{2,2} & a_{2,3} \end{vmatrix}$$

The 2 by 2 determinants are called *minors.* Every element in a determinant has a corresponding minor, formed by deleting the row and column containing that element. For a determinant of order n, the minors are of order (n-1).

In general a determinant of order n is calculated from

$$\sum_{i=1}^{n} (-1)^{i+1} \times a_{i,1} \times m_{i,1}$$

where *m* is the minor of *a*.

Cofactors

The cofactor of an element is the minor multiplied by the appropriate sign

$$c_{i,1} = (-1)^{i+1} \times m_{i,1}$$

or more generally

$$c_{i,j} = (-1)^{i+j} \times m_{i,j}$$

Adjoint Matrices

Every square matrix has an adjoint matrix, found by taking the matrix of its cofactors, and transposing it, ie if

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & . & . & a_{1,n} \\ a_{2,1} & a_{2,2} & . & . & a_{2,n} \\ . & . & & & . \\ . & . & & & . \\ a_{n,1} & a_{n,2} & . & . & a_{n,n} \end{pmatrix}$$

then the adjoint is

$$adj(A) = \begin{pmatrix} c_{1,1} & c_{2,1} & . & . & c_{n,1} \\ c_{1,2} & c_{2,2} & . & . & c_{n,2} \\ . & . & & & . \\ . & . & & & . \\ c_{1,n} & c_{2,n} & . & . & c_{n,n} \end{pmatrix}$$

Calculating the Inverse of a Matrix

After the previous slightly complex definitions, the calculation of the inverse matrix is relatively simple.

$$A^{-1} = \frac{adj(A)}{|A|}$$

Clearly, if the <u>determinant</u> of A is zero, the inverse cannot be calculated and the matrix is said to be singular.

5 Application – Solving Linear Equations

One application of matrices is in solving linear equations (or simultaneous equations as they are often known). For example:

$x + y + z = 6$

$2x + 3y + 4z = 20$

$4x + 2y + 3z = 17$

This can be written in terms of matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 20 \\ 17 \end{pmatrix}$$

or more generally

$$A \times X = R$$

To solve this we simply need to pre-multiply both sides by the <u>inverse</u> of A

$$A^{-1} \times A \times X = A^{-1} \times R$$
$$X = A^{-1} \times R$$

In this case the answer is

$$\begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 3\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 2\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 6 \\ 20 \\ 17 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Eigenvalues (http://www.euclideanspace.com/maths/algebra/matrix/functions/eigenv/index.htm)

The eigenvalues of a matrix [M] are the values of λ such that:

[M] {v} = λ{v}

where {v} = a vector

this gives:

|M - λI| = 0

where I = identity matrix

this gives:

$$\begin{vmatrix} m11 - \lambda & m12 & m13 \\ m21 & m22 - \lambda & m23 \\ m31 & m32 & m33 - \lambda \end{vmatrix} = 0$$

so

(m11- λ) (m22- λ) (m33- λ) + m12 m23 m31 + m13 m21 m32 - (m11- λ) m23 m32 - m12 m 21 (m33- λ) - m13 (m22- λ) m31 = 0

the values of λ are the eigenvalues of the matrix.