Angular Momentum

So we now want to explore the angular momentum operator and try to develop another method to solve the TISE for a central potential in spherical coordinates. We start with the definition of angular momentum:

\[ \vec{L} = \vec{r} \times \vec{\rho} \]

and recalling the way we can find this in vector notation:

\[
\begin{pmatrix}
L_x \\
L_y \\
L_z
\end{pmatrix} =
\begin{pmatrix}
\vec{e}_x & \vec{e}_y & \vec{e}_z \\
x & y & z \\
p_x & p_y & p_z
\end{pmatrix}
\]

so that:

\[
L_x = yp_z - zp_y \\
L_y = zp_x - xp_z \\
L_z = xp_y - yp_x
\]

Now, before we go into spherical coordinates, let’s investigate the commutation relations of the operators that represent these components of angular momentum:

\[
[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z]
\]

\[
= [yp_z, zp_x - xp_z] - [zp_y, zp_x - xp_z]
\]

\[
= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]
\]

Remembering that:

\[
\]

we’ll look at this one term at a time:

\[
[yp_z, zp_x] = y[p_z, zp_x] + [y, zp_x]p_z
\]

\[
= -y[zp_x, p_z] - [zp_x, y]p_z
\]

\[
= -yz[p_x, p_z] - y[z, p_z]p_x - z[p_x, y]p_z - [z, y]p_x p_z
\]

\[
= -y[z, p_z]p_x
\]

\[
= -yihp_x
\]

\[
= -ihyp_x
\]
Using the same technique, one can find that the middle two terms (in the last line of the commutator of \( L_x \) and \( L_y \)) are zero, and we are left with:

\[
[L_x, L_y] = [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]
\]

\[
= -ihyp_x + ihxp_y
\]

\[
= ih(xp_y - yp_x)
\]

\[
= ihL_z
\]

You will show explicitly in homework that:

\[
[L_y, L_z] = i\hbar L_x
\]

and

\[
[L_z, L_x] = i\hbar L_y
\]

These can be found from the first result by a cyclic permutation of the indices \( x, y, \) and \( z \). (Notice that the order from left to right is always \( x, y, z \) \( y, z, x \) \( z, x, y \): \( y \) always follows \( x \) if you start around at the beginning…)

Now, this means that the individual components of angular momentum do not commute! We cannot know more than one component of angular momentum at the same time. How about the magnitude of the angular momentum and one of the components?

\[
[L^2, L_x] = [L^2_x + L^2_y + L^2_z, L_x]
\]

\[
= [L^2_x, L_x] + [L^2_y, L_x] + [L^2_z, L_x]
\]

\[
= 0 + L_y[L_y, L_x] + L_z[L_z, L_x] + L_x[L_x, L_x]
\]

\[
= L_y(-i\hbar L_z) + (-i\hbar L_z)L_y + L_z(i\hbar L_y) + (i\hbar L_y)L_z
\]

\[
= 0
\]

and it also follows that \( L^2 \) commutes with \( L_y \) and \( L_z \). So, \( L^2 \) commutes with each component of \( L \) (separately). Again, what does this mean? It means 1) we can measure \( L^2 \) and one component of \( L \) at the same time with no limit on the uncertainty and 2) there exists simultaneous eigenfunctions of both \( L^2 \) and one of the components:

\[
L^2\psi = \Lambda^2\psi
\]

and

\[
L_x\psi = \mu\psi
\]

and our goal now is to find the eigenvalues and eigenfunctions \( \psi \) that obeys these eigenvalue equations. To find them, let’s try out the technique of ladder operators that
we used for the Harmonic oscillator. Here, I will give you no motivation for the choice of the form for the ladder operators, except as a postscript when we are done with it:

\[ L_+ = L_x + iL_y \]
\[ L_- = L_x - iL_y \]

or

\[ L_\pm = L_x \pm iL_y \]

OK, before we go exploring the properties of these ladder operators, let’s try to recall what was cool about the harmonic oscillator ladder operators. Remember that the ladder operators turned a state (or eigenfunction) into the next higher or lower state (or eigenfunction). Since both the original state and the raised (or lowered) state were both eigenfunctions of the Hamiltonian, then that meant that the ladder operator operating on an eigenfunction of the Hamiltonian was also an eigenstate of the Hamiltonian. Of course, this broke down when applying the lowering operator to the lowest state. But, ignoring that for the moment (it will be clear how this displays itself here later), we press on to see how the ladder operators for angular momentum work in our favor.

We first look at their commutation relations with the z-component of angular momentum:

\[
\begin{align*}
\{ L_z, L_\pm \} &= [L_z, L_\pm] + i[L_z, L_y] \\
&= i\hbar L_y \pm i(-i\hbar L_y) \\
&= \pm\hbar L_\pm
\end{align*}
\]

OK, what about the commutator of \( L_2 \) with the ladder operators?

\[
\begin{align*}
\{ L^2, L_\pm \} &= [L^2, L_\pm] + i[L^2, L_y] \\
&= 0 \pm i(0) \\
&= 0
\end{align*}
\]

Now, because the two operators commute, we are going to make the same claim that we did in the harmonic oscillator case, that is, that a ladder operator operating on an eigenfunction of \( L_2 \) is also an eigenfunction of \( L_2 \):

\[
L^2 \psi_\pm = \Lambda^2 \psi_\pm
\]

\[
L^2(L_\pm \psi) = L_\pm (L^2 \psi) = L_\pm (\Lambda^2 \psi) = \Lambda^2 (L_\pm \psi) \Rightarrow
\]

\[
L^2(L_\pm \psi) = \Lambda^2 (L_\pm \psi)
\]

In fact, \( L_\pm \psi \) is an eigenfunction of \( L_2 \) with the same eigenvalue as \( \psi \)!
How about $L_z$? Is $L_z \psi$ also an eigenfunction of it?

\[ L_z \psi = \mu \psi \]
\[ L_z (L_z \psi) = (L_z L_z - L_z L_z) \psi + L_z L_z \psi = [L_z, L_z] \psi + L_z L_z \psi = \pm \hbar L_z \psi + L_z \mu \psi = (\mu \pm \hbar)(L_z \psi) \]

So that $L_z \psi$ is an eigenfunction of $L_z$ with the eigenvalue increased (or decreased) by $\hbar$!

This is the reason we call these ladder operators, they change the state to one of higher (or lower) eigenvalue for the $z$-component of the angular momentum. Now, there must be limits to this increase (and decrease) since the $z$-component can’t be larger than the total angular momentum. So, let’s assume that there exists a largest value of the $z$-component eigenvalue – call it $l$. For now, let’s not put any meaning into what $l$ is. Then we know that:

\[ L_z \psi_l = \hbar l \]
\[ L_z \psi_0 = 0 \]

and

\[ L^2 \psi_l = \Lambda^2 \psi_l \]

OK now, hold on to your hats. Much of what I’m about to do has an analogy with what we did for the harmonic oscillator potential, so be looking for the similarities.

\[ L_z L_z = (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2 \mp i(L_x L_y - L_y L_x) = L_x^2 + L_y^2 \mp i[L_x, L_y] = L_x^2 + L_y^2 \mp i(\hbar L_z) = L_x^2 + L_y^2 \pm \hbar L_z = L^2 - L_z^2 \pm \hbar L_z \]

We can turn this around to get $L^2$ in terms of the other operators:

\[ L^2 = L_z L_z + L_z^2 \mp \hbar L_z \]

Then, let’s look at the eigenvalue equation for $L^2$ with the eigenfunction that corresponds to the highest value of the $z$-component:
\[ L^2 \psi_i = \left( L_+ L_+ + L_-^2 + \hbar L_z \right) \psi_i \]
\[ = L_+ L_+ \psi_i + L_-^2 \psi_i + \hbar L_z \psi_i \]
\[ = L_+ \left( L_+ \psi_i \right) + L_- \left( L_- \psi_i \right) + \hbar L_z \psi_i \]
\[ = L_+ \left( 0 \right) + L_- \left( \hbar \right) \psi_i + \hbar \left( \hbar \right) \psi_i \]
\[ = \left( \hbar \right) \left( L_+ \psi_i \right) + \hbar \left( \hbar \right) \psi_i \]
\[ = \left( \hbar \right) \left( L_- \psi_i \right) + \hbar \left( \hbar \right) \psi_i \]
\[ = \hbar^2 \left( l + 1 \right) \psi_i \Rightarrow \]
\[ \Lambda^2 = \hbar^2 \left( l + 1 \right) \]

Remember – no meaning for \( l \) just yet!!! This just tells us the eigenvalue of \( L^2 \) in terms of the \textit{maximum} eigenvalue of \( L_z \). Right now, nothing more. OK, let’s repeat this for the minimum eigenvalue of \( L_z \). We’ll assume that there is a minimum eigenvalue equation for \( L_z \) such that:

\[ L_z \psi_{\bar{I}} = \hbar \bar{I} \psi_{\bar{I}} \]
\[ L_- \psi_{\bar{I}} = 0 \]

\textit{and}

\[ L^2 \psi_{\bar{I}} = \Lambda^2 \psi_{\bar{I}} \]

Then, using the other form for \( L^2 \):

\[ L^2 \psi_{\bar{I}} = \left( L_+ L_- + L_-^2 - \hbar L_z \right) \psi_{\bar{I}} \]
\[ = L_+ L_- \psi_{\bar{I}} + L_-^2 \psi_{\bar{I}} - \hbar L_z \psi_{\bar{I}} \]
\[ = L_+ \left( L_- \psi_{\bar{I}} \right) + L_- \left( L_- \psi_{\bar{I}} \right) - \hbar L_z \psi_{\bar{I}} \]
\[ = L_+ \left( 0 \right) + L_- \left( \hbar \bar{I} \right) \psi_{\bar{I}} - \hbar \left( \hbar \bar{I} \right) \psi_{\bar{I}} \]
\[ = \left( \hbar \bar{I} \right) \left( L_- \psi_{\bar{I}} \right) - \hbar \left( \hbar \bar{I} \right) \psi_{\bar{I}} \]
\[ = \left( \hbar \bar{I} \right) \left( L_- \psi_{\bar{I}} \right) - \hbar \left( \hbar \bar{I} \right) \psi_{\bar{I}} \]
\[ = \hbar^2 \left( \bar{I} - 1 \right) \psi_{\bar{I}} \Rightarrow \]
\[ \Lambda^2 = \hbar^2 \left( \bar{I} - 1 \right) \]

Now, we have two equations for \( \Lambda^2 \):
\[ \Lambda^2 = \hbar^2 l(l-1) \]
\[ \Lambda^2 = \hbar^2 \tilde{I}(\tilde{I} - 1) \]  \[ \Rightarrow \]
\[ \hbar^2 l(l+1) = \hbar^2 \tilde{I}(\tilde{I} - 1) \]

So, either \( \tilde{I} = (l + 1) \) which doesn’t make sense since \( l \) was supposed to be the highest eigenvalue of \( L_z \), or \( \tilde{I} = -l \). So, the eigenvalues of \( L_z \) go from \(-\hbar\) to \( \hbar \). And, from what we saw with the ladder operators, they increase in units of \( \hbar \). So let’s rewrite the eigenvalue equation for \( L_z \) to reflect this:

\[ L_z \psi_i^m = \hbar m \psi_i^m, \quad m = -l, -l + 1, \ldots, 0, \ldots, l - 1, l \]

Familiar? But what about the allowed values of \( l \)? Well, there are a certain number of integer steps allowed in \( m \), in fact there are \( 2l+1 \) integer steps. Let this value be \( N \), an integer. Then,

\[ l = -l + N \Rightarrow \]
\[ 2l = N \Rightarrow \]
\[ l = \frac{N}{2} \]

So, the eigenvalue equation for \( L^2 \) can be rewritten as:

\[ L^2 \psi_i^m = \hbar^2 l(l+1) \psi_i^m, \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \]

**Note:** I have gotten the eigenvalues of \( L^2 \) and \( L_z \) without finding the eigenfunctions and only by using the canonical commutation relation between position and momentum. It’s almost like magic, but you should go through it a few times until you have a good feel for its basis.

Now, before we move on to finding the eigenfunctions (you already know what they are), I want to take some time to get a good feel for what all this means. The angular momentum in the quantum world is quantized. It can have eigenvalues of:

\[ L = \hbar \sqrt{l(l+1)} \]

where \( l \) is an integer or half-integer. You cannot know more than one component of \( L \) at the same time, and when you choose one to know, call it the z-direction, it too is quantized with allowed eigenvalues:

\[ L_z = \hbar m \quad m = -l, -l + 1, \ldots, 0, \ldots, l - 1, l \]
A picture of this situation that is commonly used is the following:

\[ L = |L| = l, \]

where \( L \) is a vector in space. In the case pictured above, \( l = 2 \), so

\[ L = h \sqrt{l(l+1)} = h \sqrt{6} = 2.45h. \]

However, we cannot know its direction, because that would imply that we know more than one component of \( L \), so the arrows in the picture above are deceiving. All we can know is one component and it has, in the case above, values of 2, 1, 0, -1, and -2.