

# Lecture 39

## (Barrier Tunneling)

Physics 262-01 Spring 2019

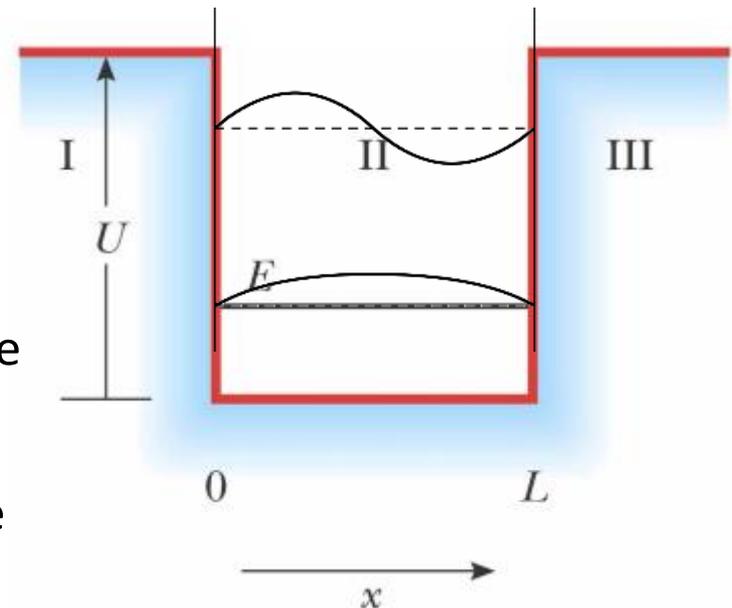
Douglas Fields

# Finite Potential Well

- What happens if, instead of infinite potential “walls”, they are finite?
- Your classical intuition will probably tell you that the wave functions look the same as for the infinite well as long as the energy is less than the height of the potential barriers, yes?
- Well, you’d be wrong. We have to solve the wave equation, and not bring our classical intuition into the problem!
- We will look at each region of space separately, starting with the region where  $V=0$ .
- There, we would expect solutions of the form we got for the infinite well:

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

- But now, the wave function does NOT have to be zero at the boundaries!
- To understand why not, we have to go back and understand why they DID for the infinite well...



# Finite Potential Well

$$E\psi(x) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x)$$

- If we look at the 1D wave equation, where  $V = \infty$ , the only way for the equation to be satisfied, is if  $\psi(x) = 0$ .
- But that is not the case when  $V$  is finite.
- To see this, for the region where  $V=U$ , let's rewrite the Schrödinger equation just a little:

$$(E - U)\psi(x) = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2}$$

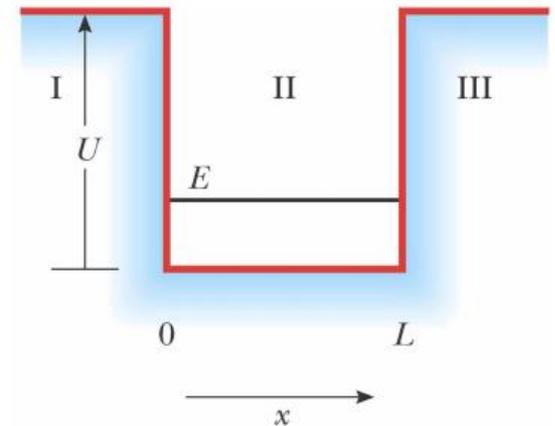
- If the energy is less than  $U$ , then the left hand side is negative, and we get for solutions:

$$\psi(x) = Ae^{\kappa x} + Be^{-\kappa x}$$

- Where the real exponents  $\kappa$  are given by:

$$\frac{\hbar^2 \kappa^2}{2m} = (U - E) \Rightarrow$$

$$\kappa = \frac{\sqrt{2m(U - E)}}{\hbar}$$



# Boundary Conditions

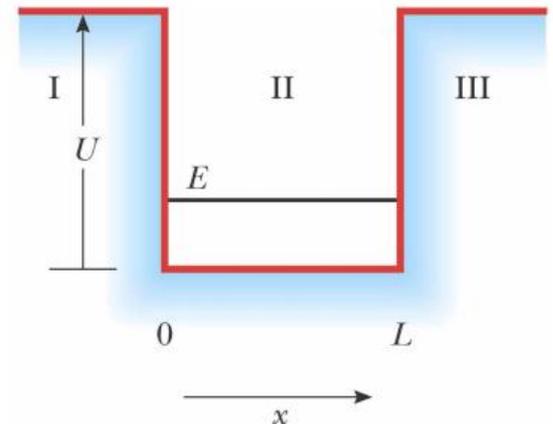
- Now, let's consider the boundary conditions of our problem.
- We know that the probability of finding the particle must approach zero outside of the well, defined by the region where  $V=0$ , so:

$$\psi(x) = Ae^{\kappa x} + Be^{-\kappa x} \Rightarrow$$

$$\text{for } x < 0, \quad B = 0$$

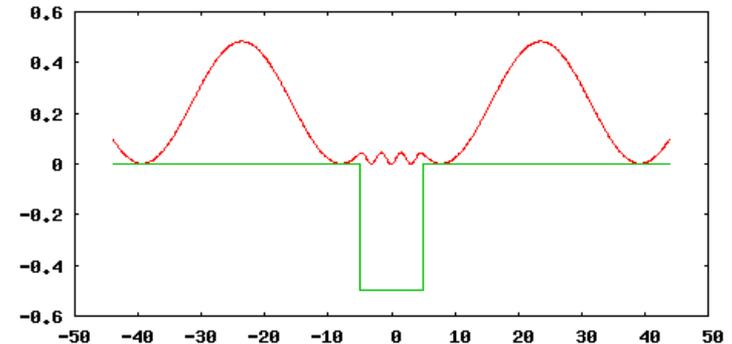
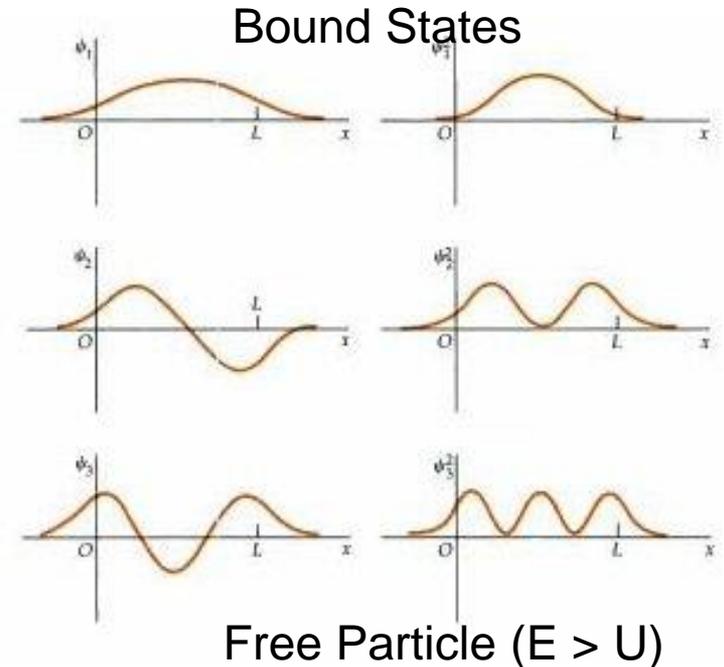
$$\text{for } x > L, \quad A = 0$$

- Or, in words, outside of the well, the wave function must be a **decaying** exponential.
- Inside the well, we have our sinusoidal functions, so we are left with just one task:
- We have to match the functions at the boundaries so that the wave function is continuous, AND
- Its first derivatives also must be continuous.
- The mathematics to do that aren't particularly difficult and are certainly not beyond your abilities, but they are also not very interesting, and you will see this again in future courses, so I won't actually solve for the wave functions, just show you what they look like:



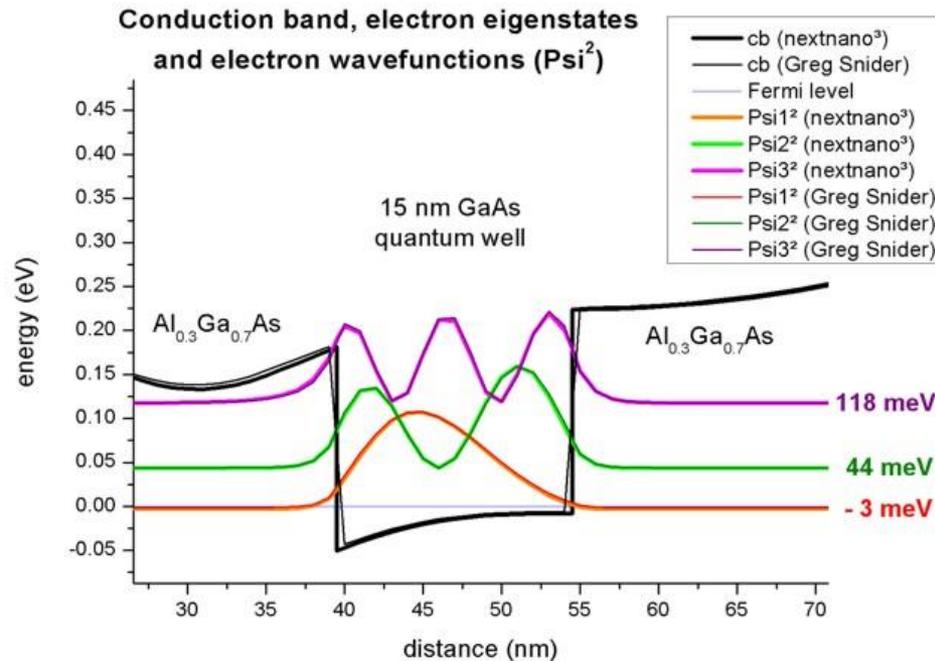
# Finite Potential Well

- The wave functions end up looking like:
- There are a few things of importance to note:
  - There are a finite number of bound states. When  $E > U$ , the particle is free:
  - The wavelengths are somewhat longer than the infinite square well since they extend beyond the well boundaries (lower energy eigenstates).
  - The wave functions extend into the classically forbidden regions!
- This last character of quantum wave functions is really important. It means that there is a finite probability of finding the particles in a region where it has a negative classical kinetic energy, and will lead to some strange behavior (or at least strange for what we think of as a particle).



# Real Example

- GaAs (semiconductor) junction with Aluminum doped GaAs.

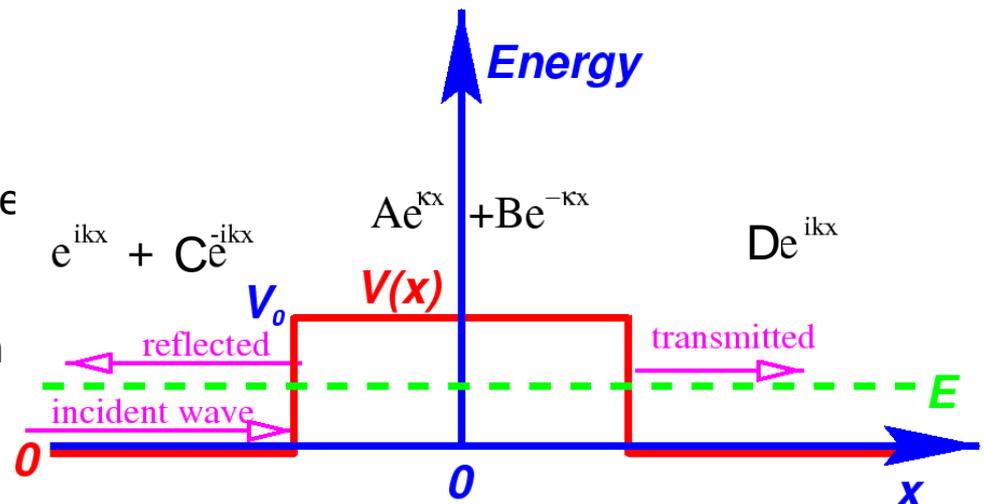


# Barrier Tunneling

- Since the wave function extends into the classically forbidden region, what happens if close to the wall, we lower the potential again?

# Barrier Tunneling

- Let's briefly examine what happens at a finite-width barrier:
- Let's first assume that we initially have a wave on the left hand side of the barrier, moving towards it to the right.
- Since  $V=0$  there, the wave function solution to the Schrödinger's equation is just  $e^{ikx}$  (normalization constant is relative, since we have to normalize over the entire  $x$ -axis).
- At the barrier,  $E < U$ , so the solutions is just  $Ae^{kx} + Be^{-kx}$ .
- On the right side of the barrier, we will have some transmitted wave  $De^{ikx}$  with a fraction of the total probability given by the normalization constant  $D^2$ .
- On the left side of the barrier, we will have some reflected wave  $Ce^{-ikx}$  with a fraction of the total probability given by the normalization constant  $C^2$ .
- All we have to do is to match the wave functions and their first derivatives at the boundaries and
- Note that the sum of  $T^2$  and  $R^2$  have to be one (the incoming wave either was transmitted or reflected).
- Solving for all of the constants can be a little messy, and not instructive at this point...



# Barrier Tunneling

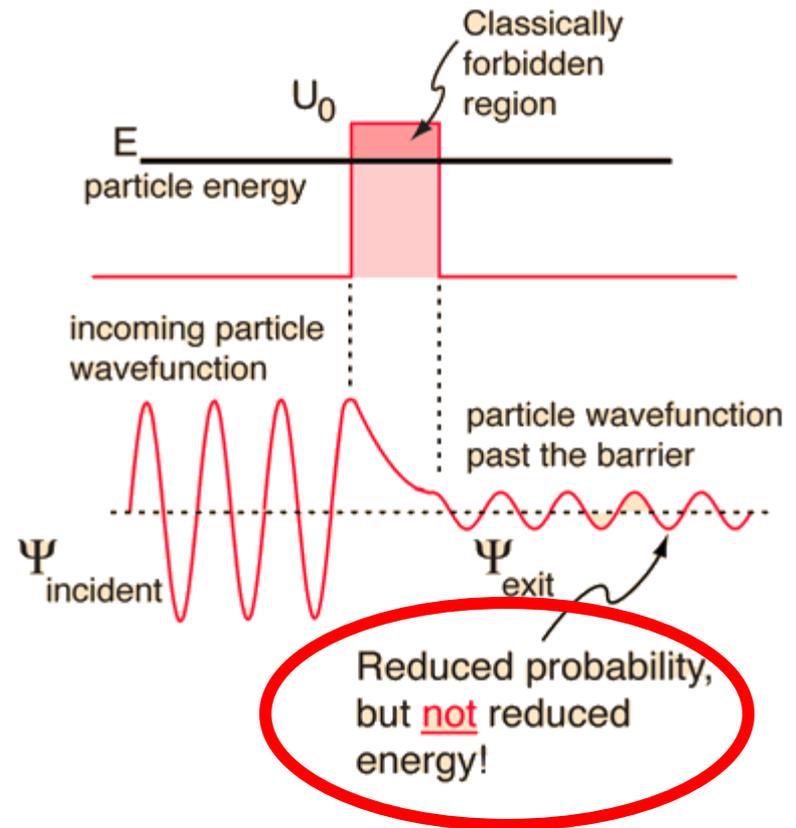
- The results look something like this (as long as the barrier is large compared to the wavelength and  $U$  is large compared to  $E$ ):
- With the constants  $T$  and  $R$  given by:

$$T = \left| \frac{D^2}{1} \right| = Ge^{-2\kappa L}, \quad R = \left| \frac{C^2}{1} \right| = 1 - T$$

$$G = 16 \frac{E}{U} \left( 1 - \frac{E}{U} \right)$$

$$\kappa = \frac{\sqrt{2m(U - E)}}{\hbar}$$

- You can solve without those conditions, but the answer is messy and difficult to interpret.



A 2.0-eV electron encounters a barrier 5.0 eV high. What is the probability that it will tunnel through the barrier if the barrier width is (a) 1.00 nm and (b) 0.50 nm?

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### SOLUTION

**IDENTIFY and SET UP:** This problem uses the ideas of tunneling through a rectangular barrier, as in Figs. 40.19 and 40.20. Our target variable is the tunneling probability  $T$  in Eq. (40.42), which we evaluate for the given values  $E = 2.0$  eV (electron energy),  $U = 5.0$  eV (barrier height),  $m = 9.11 \times 10^{-31}$  kg (mass of the electron), and  $L = 1.00$  nm or 0.50 nm (barrier width).

**EXECUTE:** First we evaluate  $G$  and  $\kappa$  in Eq. (40.42), using  $E = 2.0$  eV:

$$G = 16 \left( \frac{2.0 \text{ eV}}{5.0 \text{ eV}} \right) \left( 1 - \frac{2.0 \text{ eV}}{5.0 \text{ eV}} \right) = 3.8$$

$$U_0 - E = 5.0 \text{ eV} - 2.0 \text{ eV} = 3.0 \text{ eV} = 4.8 \times 10^{-19} \text{ J}$$

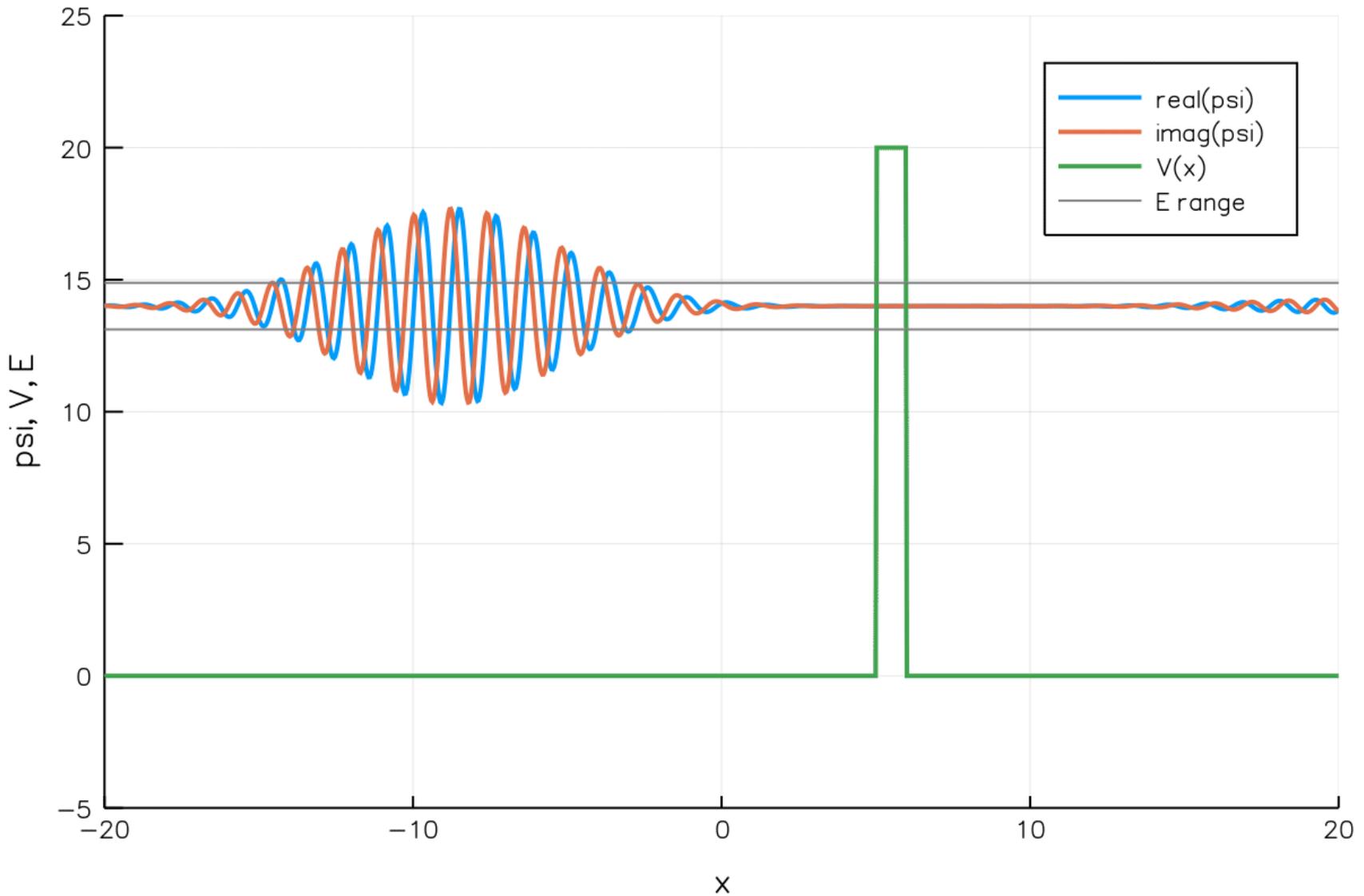
$$\kappa = \frac{\sqrt{2(9.11 \times 10^{-31} \text{ kg})(4.8 \times 10^{-19} \text{ J})}}{1.055 \times 10^{-34} \text{ J}\cdot\text{s}} = 8.9 \times 10^9 \text{ m}^{-1}$$

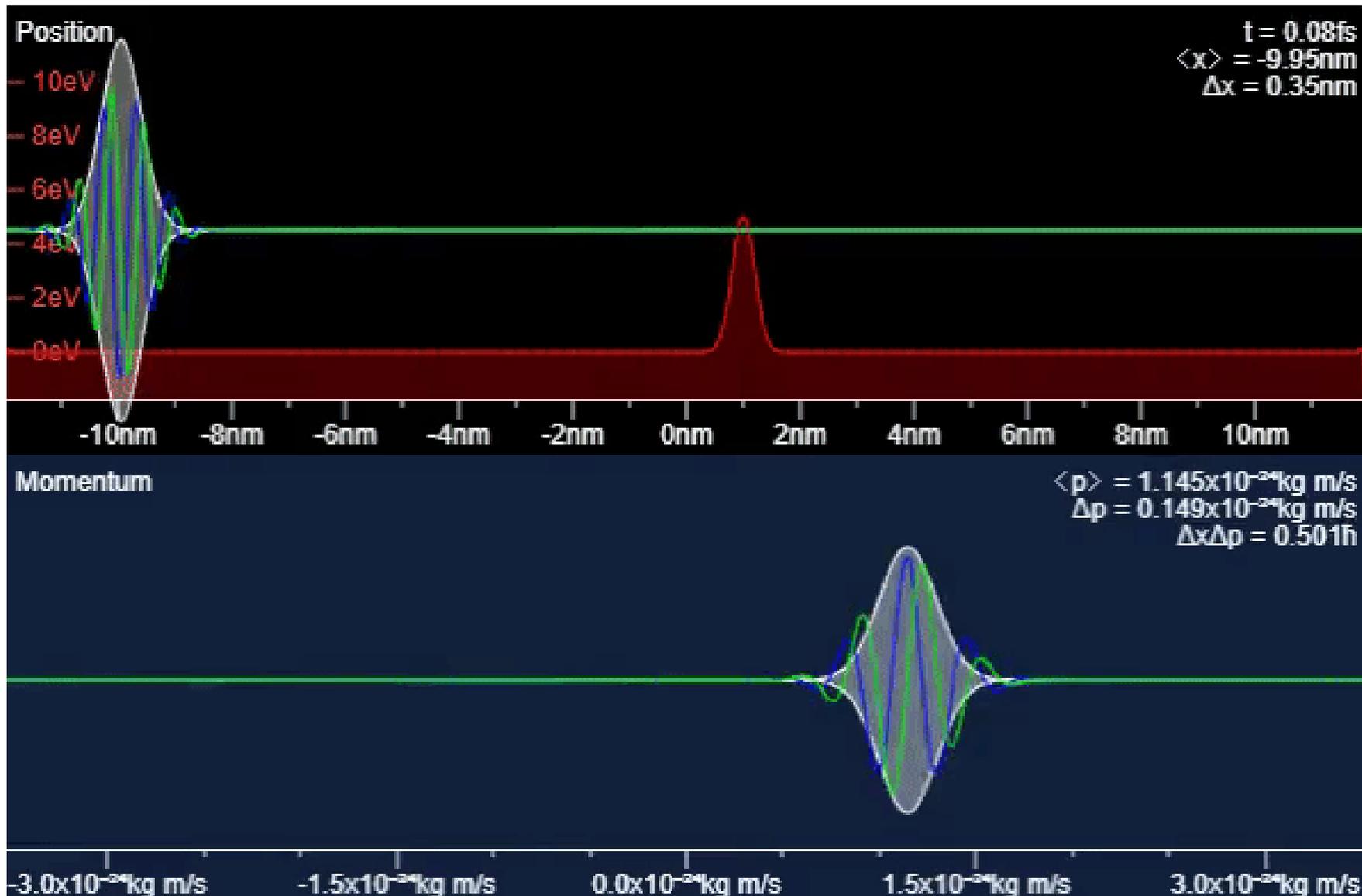
(a) When  $L = 1.00$  nm =  $1.00 \times 10^{-9}$  m,  $2\kappa L = 2(8.9 \times 10^9 \text{ m}^{-1})(1.00 \times 10^{-9} \text{ m}) = 17.8$  and  $T = Ge^{-2\kappa L} = 3.8e^{-17.8} = 7.1 \times 10^{-8}$ .

(b) When  $L = 0.50$  nm, one-half of 1.00 nm,  $2\kappa L$  is one-half of 17.8, or 8.9. Hence  $T = 3.8e^{-8.9} = 5.2 \times 10^{-4}$ .

**EVALUATE:** Halving the width of this barrier increases the tunneling probability  $T$  by a factor of  $(5.2 \times 10^{-4})/(7.1 \times 10^{-8}) = 7.3 \times 10^3$ , or nearly ten thousand. The tunneling probability is an *extremely* sensitive function of the barrier width.

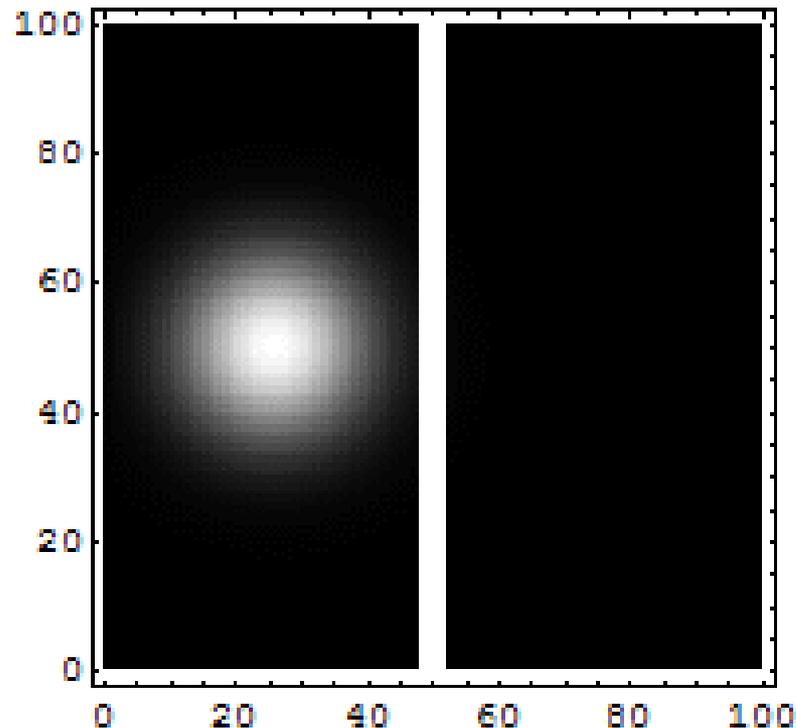
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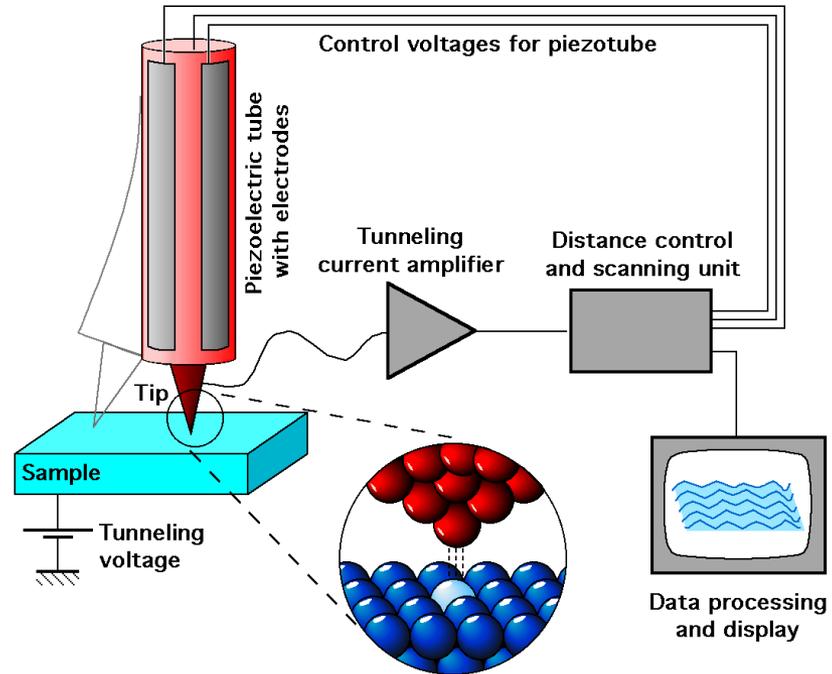
# Electron Tunneling

- Here is a simulation of an electron wave packet (probability distribution) striking a potential barrier.
- Note that even though the initial wave packet has no wave structure, when it hits the barrier and gets reflected, there are interference patterns that show up even in the probability distribution!

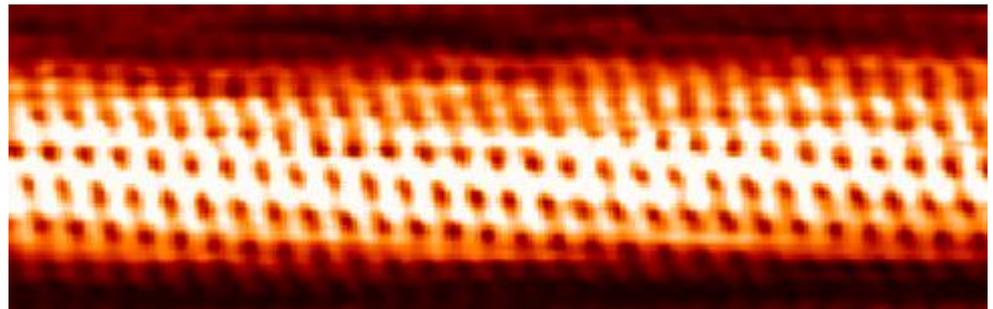


# Scanning Tunneling Microscope

- One of the aspects of science that is often overlooked is how it feeds on itself to accelerate the rate of knowledge gain.
- One of the applications of quantum tunneling is the scanning tunneling microscope.
- It uses the fact that the tunneling rate is extremely sensitive to the barrier width to get amazing 3D pictures of the microscopic world (on the atomic scale!).
- The image on the right is a STM image of a carbon nanotube.
- Maybe YOU are used to seeing things like this, but when I went to school, I was told that getting images with this kind of detail on the atomic scale was impossible...



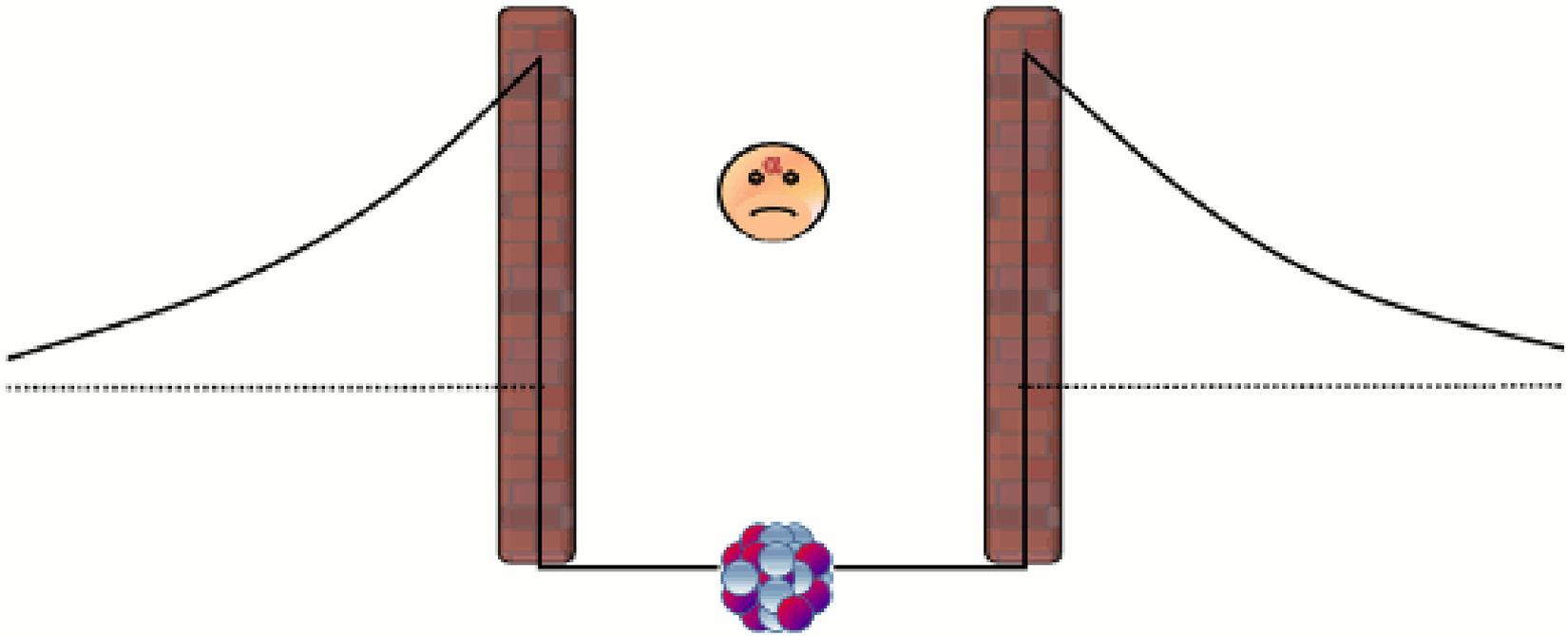
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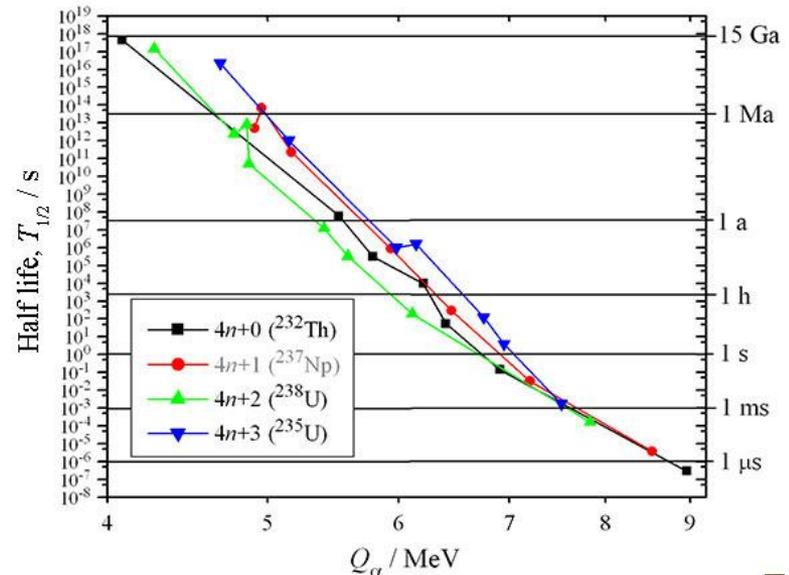
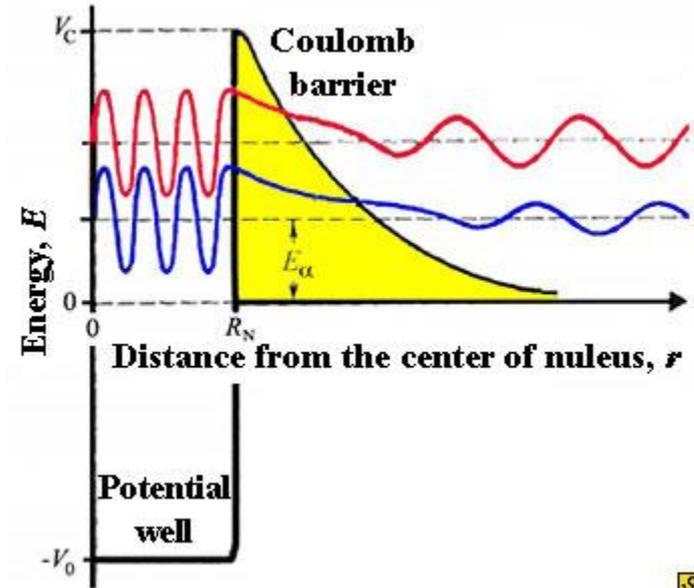
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# Alpha Decay



# Alpha Decay

- Let's consider a group of 2 protons and 2 neutrons (an alpha particle) inside a heavy nucleus.
- The alpha particle feels a strong attractive force (the strong force) inside the nucleus, but would feel a repulsive force outside (the Coulomb repulsion).
- This repulsive barrier falls as  $1/r$  outside, and hence, when the alpha has higher energy, the barrier width is lower, hence the correlation between alpha energy and the half life of the alpha decay.
- There is another reason for this – at higher alpha energies, the frequency that the alpha strikes the barrier increases as well.



# Infinite potential well with a barrier in the middle

