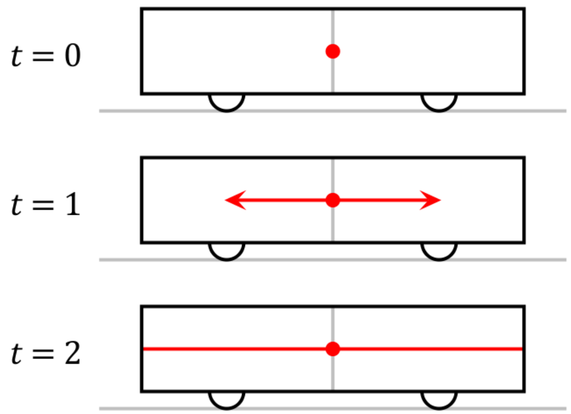
Lecture 29
(Lorentz Transformations)

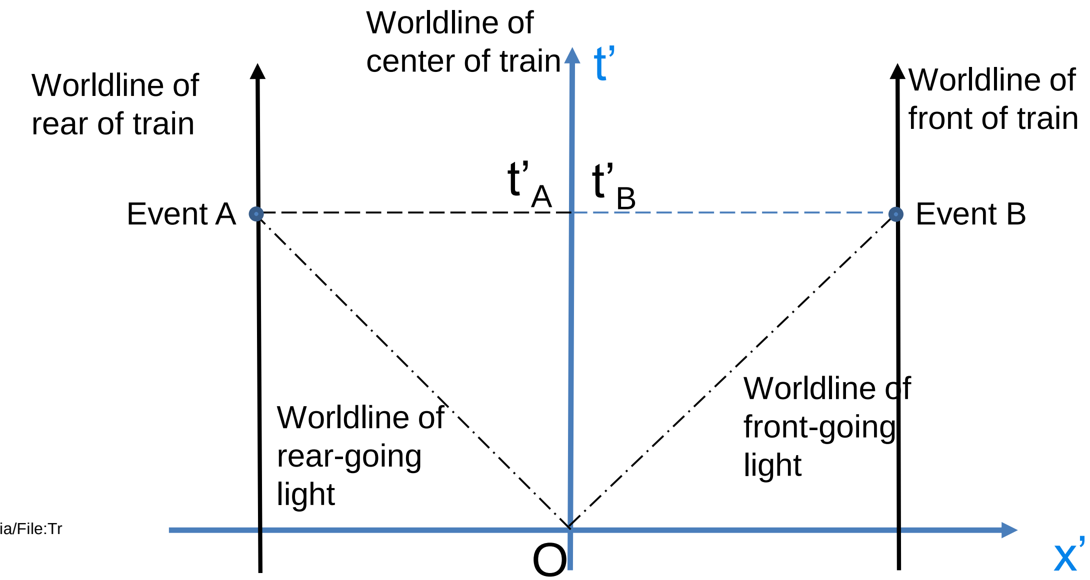
Physics 262-01 Spring 2019

Douglas Fields

Coordinate Time

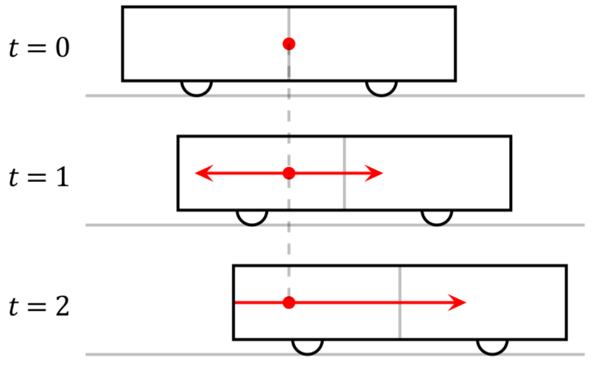
- Coordinate time is just the time difference between two events (one might, or might not occur at t = 0) as measured in one particular inertial frame.
- Let's consider a particular case: a moving train has a radar system located in the center of the train.
- It fires out two light beams at t' = 0, one towards the front, and one towards the rear.
- In the t' frame, since light moves at velocity c, the events that mark the arrival of the light beam at the front (Event B) and rear (Event A) of the train happen at the same coordinate time (relative to the origin).
- The Coordinate Time for these events is Δt' = 0 in the train frame.





Coordinate Time *is Frame Dependent*

- Now, let's look at the same set of events in the frame of the train station.
- Since light travels at v = c in the x-t frame also, the worldlines of the flashes are still at 45 degrees in that frame (v=1), but the worldlines of the train (rear, center and front) are now tilted…
- So, in the x-t frame, the events that correspond to the intersection of the worldlines of the front and rear of the train with the worldlines of the light rays no longer have the same coordinate time!
- The Coordinate Time for these events is $\Delta t \neq 0$ in the station frame.



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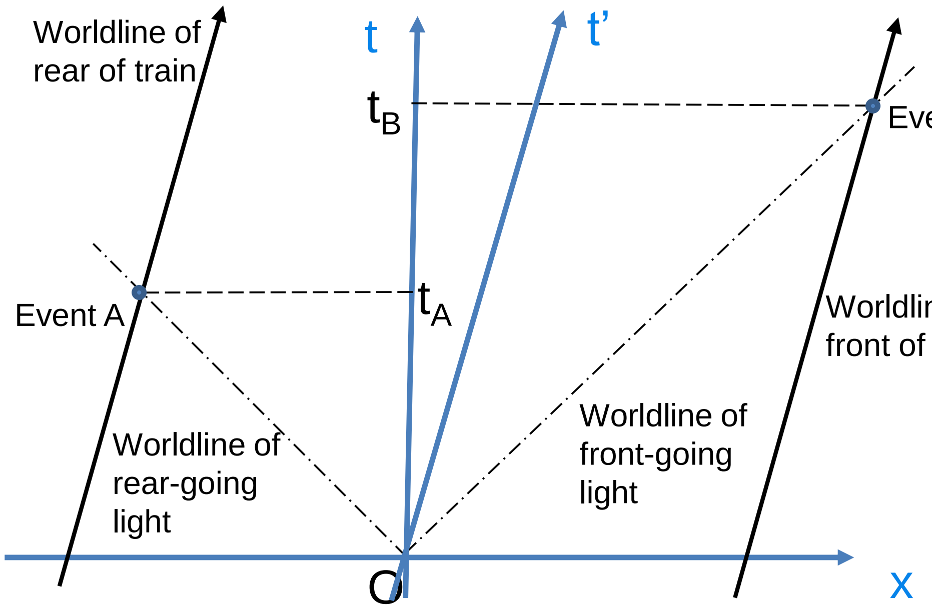
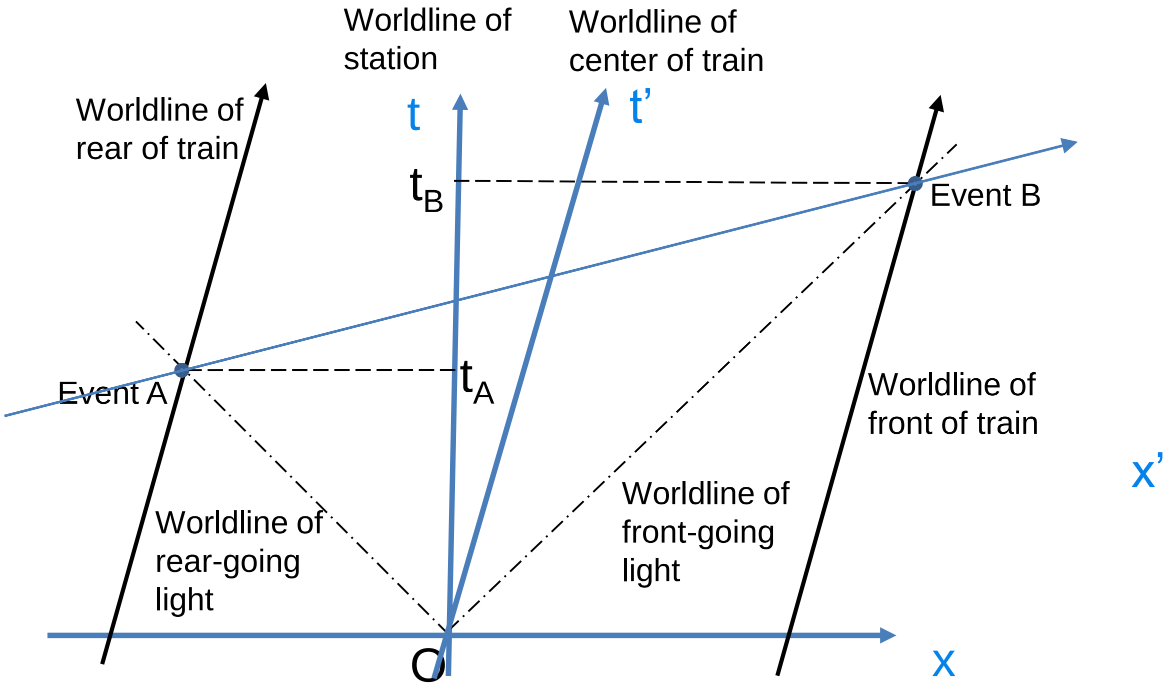


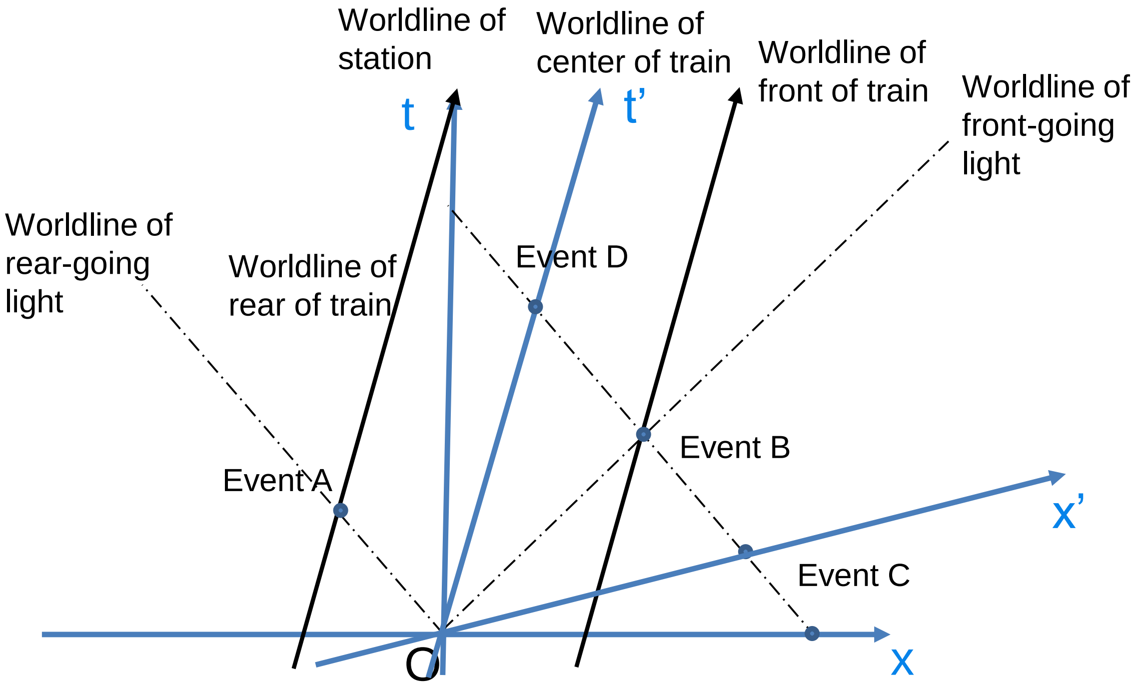
Simultaneity in the Moving Frame

- Now, if we want to represent an axis for the moving frame which represents simultaneous events in that frame, we now have two events which do exactly that!
- These events occur at the same time in the primed frame, so the line that connects them represents a set of events all with the same time.
- A line parallel to this line, but passing through the origin would then represent all events with t' = 0 – the x' axis.



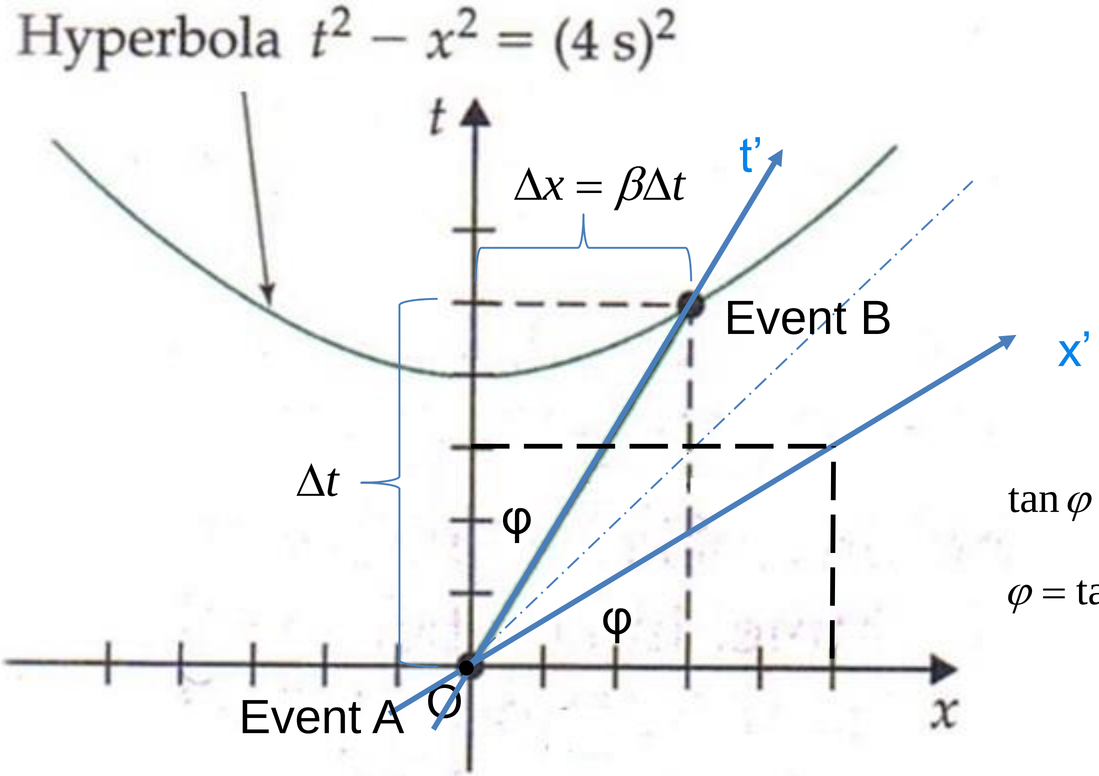
Angle of x’ axis

- Now, at t=0, let’s say another light flash occurs somewhere such that it arrives at the front of the train at exactly the same time as the light from the center of the train.
- The light flash passes the x’ axis at Event C and goes back (in both x and x’, of course), until it reaches x’=0, the t’ axis at Event D.
- Since, in the primed frame, light travels at 1, then x’ of Event C must be equal to t’ of Event D!
- So, triangles OBD and OBC must be identical, and the angle of the x’ axis above the x axis must be the same as the angle of the t’ axis to the t axis.



Two-Observer Coordinate Angles

- Alright, let's start off by thinking of some moving frame, moving relative to another inertial frame at speed β (in the diagram below, β = 3/5.
- Let there be two events (A and B) which happen at the same location in the moving frame, but separated in time by Δt'.
- The angles of the coordinate axes are just given by the inverse tangent of the relative velocity.



$$\tan\varphi = \frac{\Delta x}{\Delta t} = \frac{\beta \Delta t}{\Delta t} \Rightarrow \beta$$

$$\varphi = \tan^{-1}\beta$$

Scale Factors

- Alright, let's calculate the spacetime interval in both frames and set them equal (since the spacetime interval is invariant).
- This gives us the conversion factor between the two frames FOR EVENTS ON AXIS!!! - The scale factors.

$\Delta \equiv$ Event B - Event A

$$\Delta s' = \sqrt{(\Delta t')^2 - (\Delta x')^2} = \Delta t'$$

$$\Delta s = \sqrt{(\Delta t)^2 - (\Delta x)^2} = \sqrt{(\Delta t)^2 - (\beta \Delta t)^2} = \Delta t\sqrt{1-\beta^2}$$

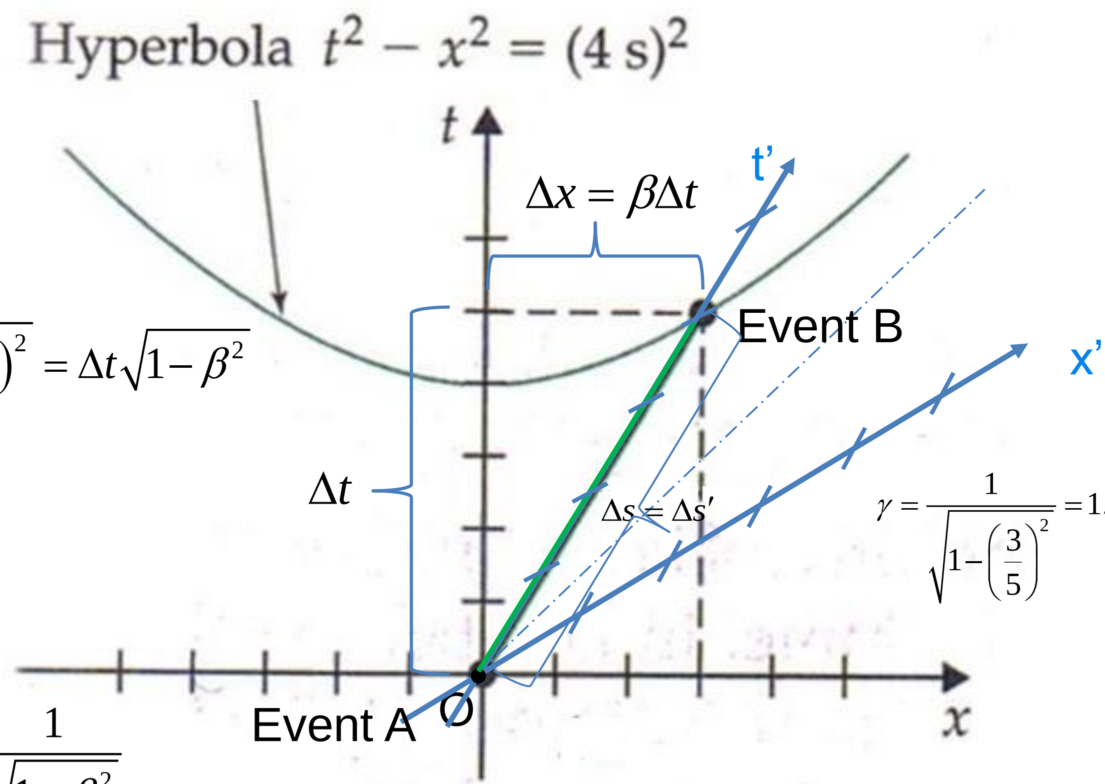
but

$$\Delta s = \Delta s' \Rightarrow$$

$$\Delta t\sqrt{1-\beta^2} = \Delta t'$$

or,

$$\Delta t = \frac{1}{\sqrt{1-\beta^2}}\Delta t' = \gamma \Delta t', \quad where\ \gamma = \frac{1}{\sqrt{1-\beta^2}}$$



$$\gamma = \frac{1}{\sqrt{1-\left(\frac{3}{5}\right)^2}} = 1.25$$

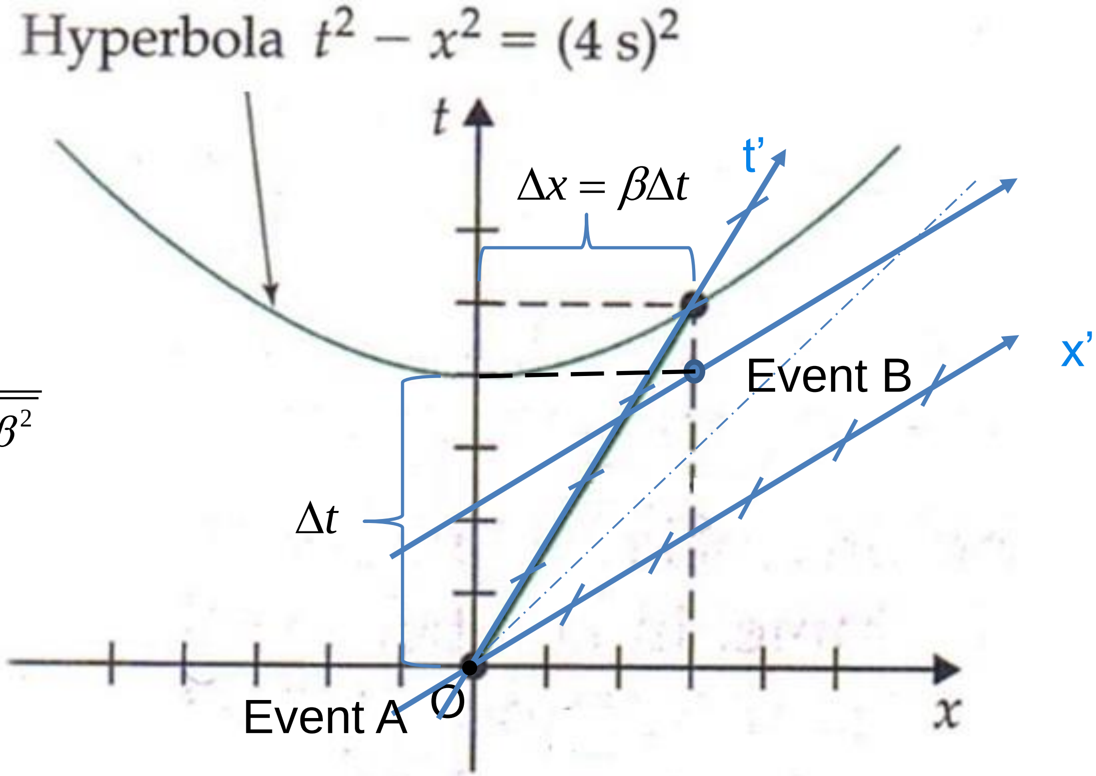
Scale Factors

- But what about events that aren't on the axis?
- For instance, let's look at the event at t=4s, x=3s.
- The conversion factor we have now would say that Δt' = 3.2s, but graphically, it looks like less that 3s!
- This is because we have to also take into account the positions in the other frames…

$$\Delta t' = \frac{1}{\gamma}\Delta t$$

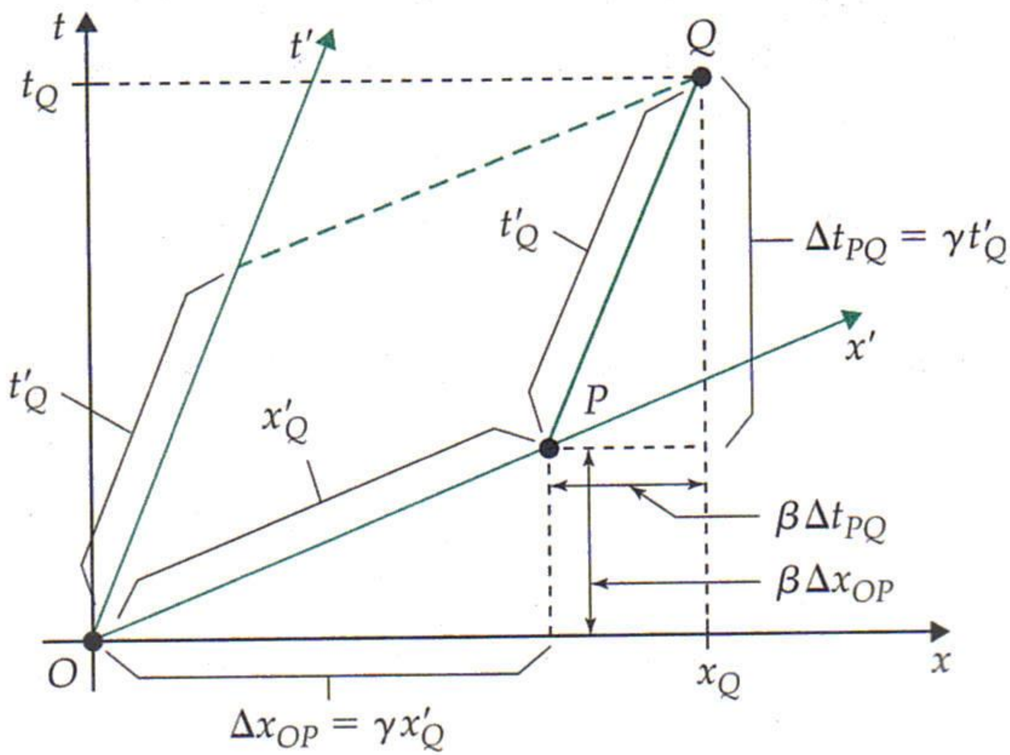
$$, \quad where\ \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\Delta t = \gamma \Delta t'$$



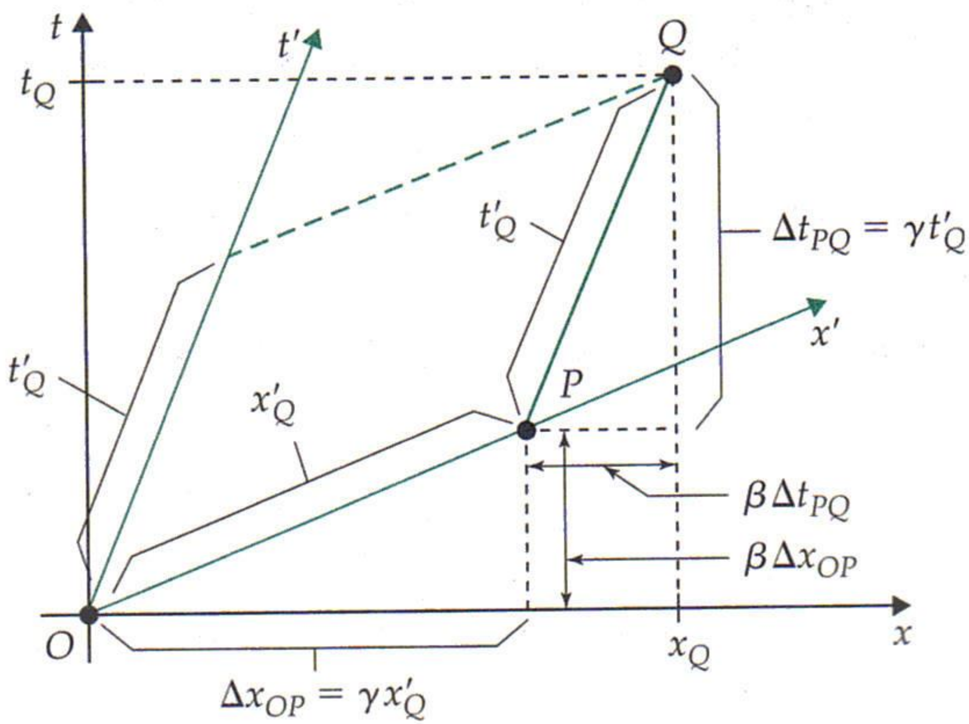
Lorentz Transformations

- Let’s look at some random event Q, not on any axis (Home or Other frames)and find its spacetime coordinates in both frames.
- First, we find an event P that occurs at the same spatial location as Q, but at t’ = 0 (on the x’ axis).
- Then, we could redefine the origin to be at P, and then Q would be on the t’ axis.
- Then the relationship between the time between P and Q in both frames is set by our scale factor.
- And the spacetime coordinates of event P in both frames is well understood.
- So we just have to sum the different terms…



Lorentz Transformations

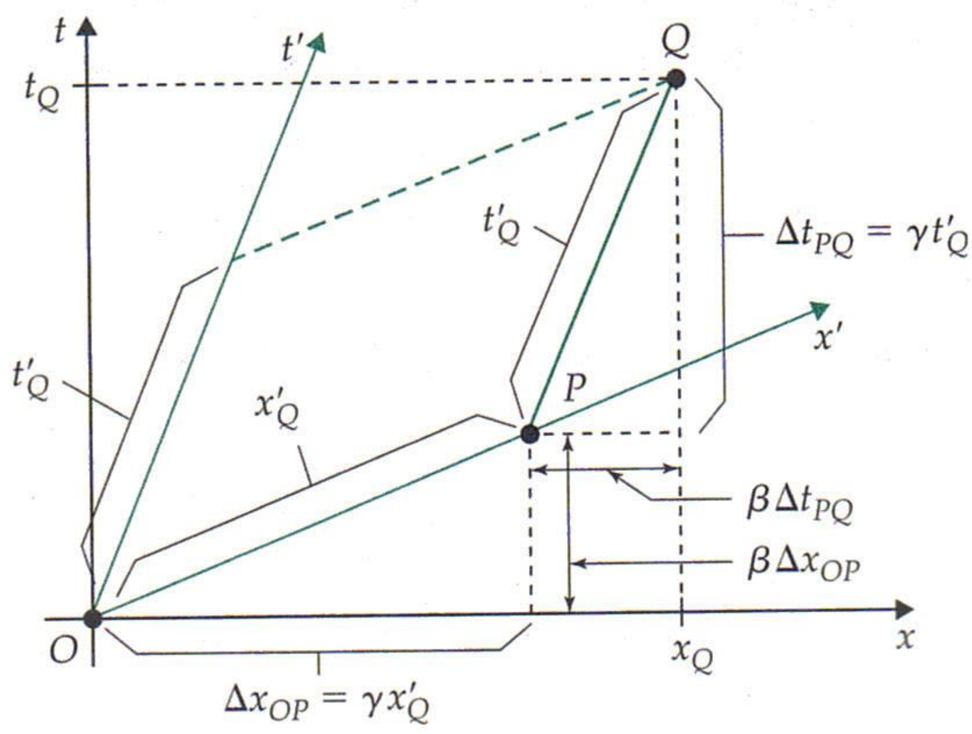
$$t_Q = \Delta t_{PQ} + \beta \Delta x_{OP}$$
$$x_Q = \Delta x_{OP} + \beta \Delta t_{PQ}$$
$but,$
$$\Delta x_{OP} = \gamma x'_Q$$
and
$$\Delta t_{PQ} = \gamma t'_Q$$
$so,$
$$t_Q = \gamma t'_Q + \beta\gamma x'_Q = \gamma\left(t'_Q + \beta x'_Q\right)$$
$$x_Q = \gamma x'_Q + \beta\gamma t'_Q = \gamma\left(x'_Q + \beta t'_Q\right)$$



Lorentz Transformations

- So, including the trivial transformations in the directions perpendicular to the relative motion between the frames, we have the correct version of the transformation equations between any two inertial frames:

$$t = \gamma\left(t' + \beta x'\right)$$
$$x = \gamma\left(x' + \beta t'\right)$$
$$y = y'$$
$$z = z'$$



Inverse Lorentz Transformations

- What about when we know t, and want to find t'?
- Let's solve the equations for t' and x':

$$t' = \frac{t}{\gamma} - \beta\left(\frac{x}{\gamma} - \beta t'\right) = \frac{t}{\gamma} - \frac{\beta x}{\gamma} + \beta^2 t' \Rightarrow$$

$$t' - \beta^2 t' = \frac{1}{\gamma}(t - \beta x) \Rightarrow$$

$$t'\left(1 - \beta^2\right) = \frac{1}{\gamma}(t - \beta x) \Rightarrow$$

$$t'\frac{1}{\gamma^2} = \frac{1}{\gamma}(t - \beta x) \Rightarrow$$

$$\boxed{t' = \gamma\left(t - \beta x\right)}$$

$$x = \gamma\left(x' + \beta t'\right) \Rightarrow$$

$$x' = \frac{x}{\gamma} - \beta t' \Rightarrow$$

$$x' = \frac{x}{\gamma} - \beta\left(\gamma\left(t - \beta x\right)\right) \Rightarrow$$

$$x' = \frac{x}{\gamma} - \beta\gamma t + \beta^2\gamma x \Rightarrow$$

$$x' = \gamma\left(x\left(\frac{1}{\gamma^2} + \beta^2\right) - \beta t\right) \Rightarrow$$

$$x' = \gamma\left(x\left(1 - \beta^2 + \beta^2\right) - \beta t\right) \Rightarrow$$

$$\boxed{x' = \gamma\left(x - \beta t\right)}$$

Symmetry is Beautiful

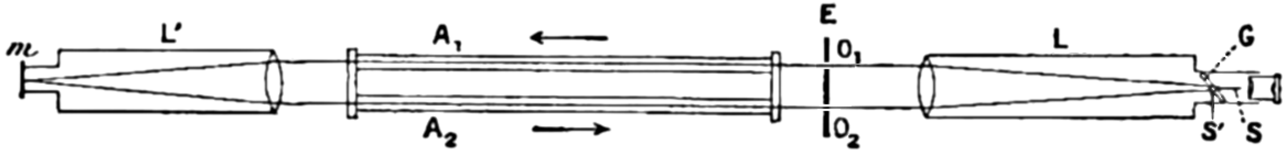
- Look at the symmetry in these equations:
  - Time and Space are symmetric
  - The conversion between frames is symmetric
- The only thing that breaks a true symmetry between time and space is the one thing that breaks symmetry in space – the velocity β.

$$\boxed{\begin{aligned} t &= \gamma\left(t' + \beta x'\right) \\ x &= \gamma\left(x' + \beta t'\right) \\ y &= y' \\ z &= z' \end{aligned}}$$

$$\boxed{\begin{aligned} t' &= \gamma\left(t - \beta x\right) \\ x' &= \gamma\left(x - \beta t\right) \\ y' &= y \\ z' &= z \end{aligned}}$$

The Whole Basis for the Derivation

- One might think that the derivation of these transformation equations was long, convoluted and not very inspiring…
- But remember that Lorentz derived these transformations (1895) long before Einstein came up with the theory of Special Relativity (1905)!
- They were derived by Lorentz to explain experiments that showed that the speed of light didn’t depend on the speed of the medium through which it was travelling.
- These experiments preceded the Michelson Morley experiment (1857) by many years…



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- The Fizeau experiment tested to see if light propagating through moving water travelled at a speed different to what it would travel if the water didn’t move.
- Like the Michelson Morley experiment, they found little difference, but this experiment took place in 1851.
- But wait! François Arago started it all with a measurement of the refractive angle of light from stars in 1810, also showing no effect.
- The difference is, Lorentz and the rest believed that this was because the ether was being dragged along by matter…

An aside and clarification…

- Remember when I talked about experiments that measured the speed of light?
- There was one that looked at the speed of light through moving water to see if there was a difference in the speed when the water was moving in the same or opposite directions as light.
- It was found that it didn't matter, except for a small deviation (much less than the water speed).
- First, the clarification: the speed of light is different in water than in vacuum.
- Also, the speed that we are referring to is the phase velocity:

$$v_{\mathrm{p}} = \frac{c}{n}$$

- The small deviation is from the (small) Doppler shifting of the wavelength of the light as viewed by the water, since the index of refraction depends, in general, on the wavelength.
- The group velocity also depends on the dispersion, but in a more complicated way…

$$v_g = c\left(n - \lambda\frac{dn}{d\lambda}\right)^{-1} = v_p\left(1 - (\lambda / n)\frac{dn}{d\lambda}\right)^{-1}$$

Einstein and Lorentz

- It will be clear by what has been said that the impressions received by the two observers A0 and A would be alike in all respects. It would be impossible to decide which of them moves or stands still with respect to the ether, and there would be no reason for preferring the times and lengths measured by the one to those determined by the other, nor for saying that either of them is in possession of the "true" times or the "true" lengths. This is a point which Einstein has laid particular stress on, in a theory in which he starts from what he calls the principle of relativity, [...] I cannot speak here of the many highly interesting applications which Einstein has made of this principle. His results concerning electromagnetic and optical phenomena ... agree in the main with those which we have obtained in the preceding pages, ***the chief difference being that Einstein simply postulates what we have deduced, with some difficulty and not altogether satisfactorily, from the fundamental equations of the electromagnetic field. By doing so, he may certainly take credit for making us see in the negative result of experiments like those of Michelson, Rayleigh and Brace, not a fortuitous compensation of opposing effects, but the manifestation of a general and fundamental principle.*** [...] It would be unjust not to add that, besides the fascinating boldness of its starting point, Einstein's theory has another marked advantage over mine. Whereas I have not been able to obtain for the equations referred to moving axes *exactly* the same form as for those which apply to a stationary system, Einstein has accomplished this by means of a system of new variables slightly different from those which I have introduced.

Symmetry from Spacetime Diagrams

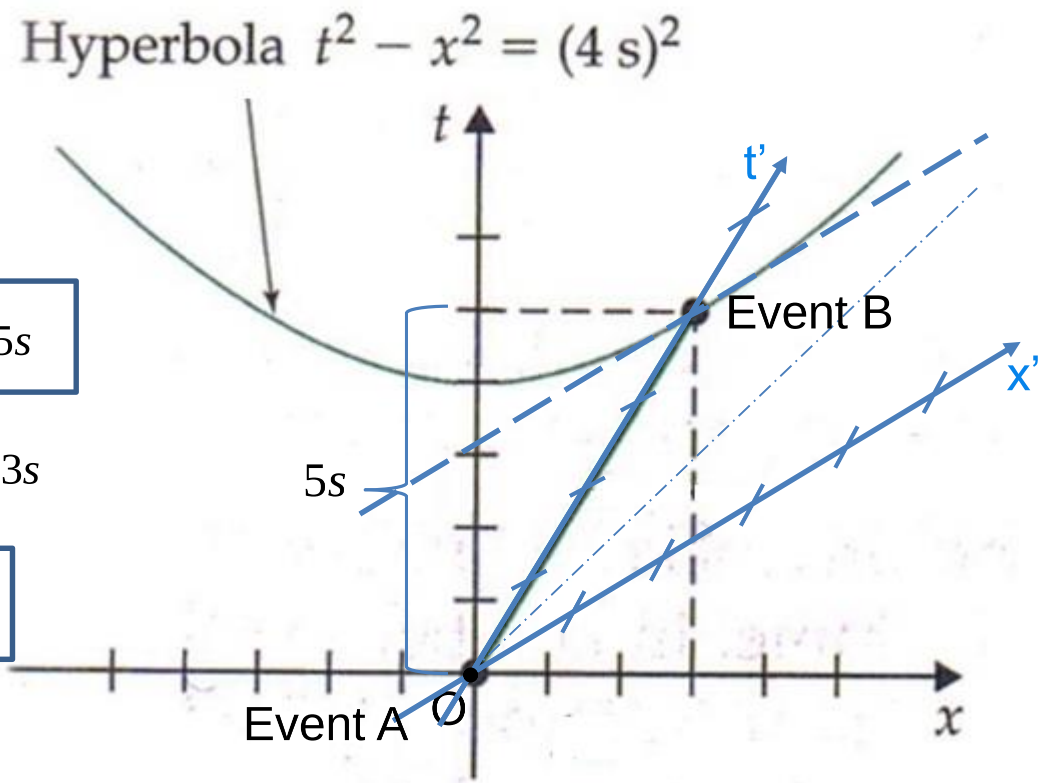
- Let's look again at our spacetime diagram we had earlier.
- From the Home Frame observation of Event B, the time in their Frame is 5s, and the time in the Other Frame is 4s.
- But, from the Other Frame's perspective, they measure 4s in their frame, and see that simultaneous with that is just over 3s in the Home Frame!
- But, wait…

$$t = \gamma(t' + \beta x') = 1.25\left(4s + \frac{3}{5}\cdot 0s\right) = 5s$$

$$x = \gamma(x' + \beta t') = 1.25\left(0s + \frac{3}{5}\cdot 4s\right) = 3s$$

$$t' = \gamma(t - \beta x) = 1.25\left(5s - \frac{3}{5}3s\right) = 4s$$

$$x' = \gamma(x - \beta t) = 1.25\left(3s - \frac{3}{5}5s\right) = 0s$$



Symmetry from Spacetime Diagrams

- Then, how do we see that? It's nowhere in the math…
- The problem is, we need to ask the question appropriately:
- What is the time of Event C, which is at x=0 in the Home frame, and is simultaneous with Event B as observed in the primed frame?

$$t' = 4s$$
$$x = 0$$
$$t = \gamma(t' + \beta x')$$
$$but$$
$$x' = -\beta t' \Rightarrow$$
$$t = \gamma(t' + \beta x') = \gamma(t' - \beta^2 t') = \gamma t'(1 - \beta^2) = \frac{t'}{\gamma}$$
$$= \frac{4s}{1.25} = 3.2s$$